Asymptotic Behavior of Conformal Metrics on Riemann Surfaces with Negative Curvatures near Isolated Singularities

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DISSERTATION

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Chapter 1

Introduction

The research of conformal metrics has a long history, since the time of Liouville [22] and Picard [26, 27]. It is recognized that conformal metrics are “ubiquitous” in complex analysis (see, for instance, Ahlfors [3], Heins [11], Kraus and Roth [17]). The class of conformal metrics with negative curvatures, such as SK-metrics and Kobayashi metrics, plays an important role in non-Euclidean geometry, from the viewpoint of Geometric Function Theory (GFT) and the theory of partial differential equations (PDEs). It also connects Riemann surfaces with differential geometry.

The starting point of our investigation is the well-known *uniformisation theorem.* For an arbitrary Riemann surface X, the universal covering space $\widetilde{X}$ is homoeomorphic, by a conformal mapping π, to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. These three standard surfaces have their own geometry, which can be described in terms of the spherical, Euclidean and hyperbolic metric, respectively. The covering group G of $\widetilde{X}$ is a subgroup of the isometry group on $\widetilde{X}$, and G is discrete and torsion-free (see, e.g. [16, Chapter 4]). Hence, the conformal mapping π induces the corresponding metric λ_X on X from the metric $\lambda_{\widetilde{X}}$ on $\widetilde{X}$, $\lambda_X := \pi^*\lambda_{\widetilde{X}}$. Specifically, if $\widetilde{X} = \hat{\mathbb{C}}$, X is conformally equivalent to $\hat{\mathbb{C}}$, and the covering group G is trivial. If $\widetilde{X} = \mathbb{C}$, X is conformally equivalent to $\mathbb{C}$ (with G trivial), the once-punctured plane or a complex torus (with G being an abelian group of rank two). If $\widetilde{X} = \mathbb{D}$, $G \subseteq \mathcal{M}$, where $\mathcal{M}$ is the group of Möbius transformations preserving $\mathbb{D}$, and X is *hyperbolic*, λ_X is the *hyperbolic metric on* X. In this case, when G is non-elementary, π is hard to find and so the hyperbolic metric is hard to give explicitly. Many estimates for the hyperbolic metric on various domains were given (see, e.g. [7], [13], [15], [31], and [16, Chapter 14]). The hyperbolic metric on X has a characterization expressed by the Gaussian curvature κ_λ: A conformal metric $\lambda(z)|dz|$ on X is the hyperbolic metric if and only if it is complete and $\kappa_\lambda \equiv -4$. The necessity is trivial and the sufficiency is implied by Yau's maximal principle (see [33]).

There are two directions to generalize the hyperbolic metric $\lambda(z)|dz|$ on a surface X. The first one is to generalize the Gaussian curvature κ_λ. The SK-metric is such a generalization. The concept of SK-metrics has its root in Ahlfors' 1938 paper [3], and the definition was given by Heins [11]. A basic property of SK-metrics is that, the hyperbolic metric is the unique maximal SK-metric on a hyperbolic domain, which is an extension of Schwarz's lemma by Ahlfors. The terminology "SK" means that "the curvature is subordinate to -4". The SK-metric has a tight connection with the subharmonic function on a domain. For a given Riemann surface and a finite number of points, McOwen [23] proved the existence and uniqueness of the hyperbolic metric with prescribed singularities at the given points. Troyanov [32] discussed conditions under which a function on a Riemann surface is the Gaussian curvature of some conformal metric with a prescribed set of singularities of conical types. By applying potential theory, Kraus and Roth [19] estimated the asymptotic behavior of the solutions to the curvature equation $\Delta u = -\kappa(z)e^{2u}$ with strictly negative, Hölder continuous Gaussian curvature function $\kappa(z)$. This curvature equation has an exquisite geometric interpretation: every solution of it gives rise to a conformal metric $e^{u(z)}|dz|(= \lambda(z)|dz|)$ with Gaussian curvature $\kappa(z)$ and vice versa (see [19]). Potential theory originates from the mathematical physics in the 19th century, and has proved to be useful in the broad range of analysis. It is easy to find the close connection between potential theory and complex analysis, and many theorems in potential theory have a variety of applications in complex analysis (see [28]).

The other way is that we allow the hyperbolic metric $\lambda(z)|dz|$ to have isolated singularities of orders no greater than 1. In the case we call $\lambda(z)|dz|$ the *generalized hyperbolic metric*, since we get back the standard hyperbolic metric when there is no singularity. In 1853 Liouville [22] obtained a general expression of the hyperbolic metric on the punctured unit disk, based on the Liouville equation $\Delta u = 4e^{2u}$. Nitsche [25] described the behavior of real-valued solutions to the the Liouville equation. When the Riemann sphere has three singularities, it is the maximal hyperbolic surface by the uniformisation theorem. Using a Möbius transformation, we may assume the three singularities are 0, 1 and ∞ of order α, β, γ, respectively, and denote the generalized hyperbolic metric on the thrice-punctured Riemann sphere $\widehat{\mathbb{C}}\backslash\{0, 1, \infty\}$ by $\lambda_{\alpha,\beta,\gamma}(z)|dz|$. Useful formulas for $\lambda_{1,1,\gamma}(z)|dz|$ were obtained by Agard [2] for $\gamma = 1$, and then by Anderson, Sugawa, Vamanamurthy and Vuorinen [6] for $\gamma \in (0, 1]$. Hempel [12] proved a sharp, explicit and easy-to-use lower bound for the standard hyperbolic density $\lambda_{1,1,1}(z)$. Kraus, Roth and Sugawa [21] gave the explicit formula $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ for the generalized hyperbolic metric on the thrice-punctured Riemann sphere $\widehat{\mathbb{C}}\backslash\{0, 1, \infty\}$, when $\alpha + \beta + \gamma > 2$. They also gave several limits for its Schwarzian near the origin. Their results are practical and enlightening to find the formula of the hyperbolic metrics and to consider the uniformisation problem for various domains. (See e.g. [8, Chapter 15] for more discussion about $\lambda_{\alpha,\beta,\gamma}(z)|dz|$.)

In this thesis we investigate the asymptotic behavior of conformal metrics with negative curvatures near isolated singularities by estimating higher order derivatives of the remainder functions of metrics. We only consider the asymptotic properties near the origin in the punctured unit disk $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$. In Chapter 2 we give a short survey of the hyperbolic metric, the SK-metric, and give foundations of hypergeometric functions and Gamma functions which will be used later. As our main tool to estimate the asymptotic behavior, the logarithmic potential is also introduced in the last section. A special version of regularity theorem is given.

In Chapter 3 we give estimates of higher order derivatives for the remainder function of $\log \lambda(z)$, when the Gaussian curvature κ_λ is negative, locally Hölder continuous in $\mathbb{D}$. When the density function $\lambda(z)$ is smooth enough, the global properties of the conformal metric $\lambda(z)|dz|$ are well studied. However, near the singularities, the asymptotic behavior of $\lambda(z)|dz|$ is not very clear. We will investigate the local terms of $\lambda(z)$ near the singularities in this chapter.

In Chapter 4 we verify sharpness of the estimates given in Chapter 3. The sharpness can be verified by the standard hyperbolic metric on $\mathbb{D}^*$ given in Theorem A when $0 < \alpha < 1$. However, the remainder function is identically 0 when $\alpha = 1$. So the generalized hyperbolic metric $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ on $\widehat{\mathbb{C}}\backslash\{0, 1, \infty\}$ makes a convictive case here. Other interesting properties of $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ are also presented in this chapter.

We devote the last chapter to Minda's limits of SK-metrics. Minda [24] investigated the behavior of the density of the hyperbolic metric in a neighborhood of a puncture on the plane domain using the uniformisation theorem. His method offers us a way to describe the asymptotic behavior on an arbitrary hyperbolic region. Our results are the extension of Minda's work and we give some limits for the higher order derivatives of SK-metrics.

Chapter 2

Preliminaries

We introduce definitions and known facts of metrics and the Gaussian curvature in the first section, including the conformal invariance of the Gaussian curvature and the maximum principle for SK-metrics. We deal with special functions in the second section. We give the special values, asymptotic behavior and differential expressions of hypergeometric functions and Gamma functions. The last section is devoted to the logarithmic potentials which is the main tool of our investigations. We show the higher order derivative formula for the logarithmic potential, and list the variants of the standard regulation theorem and Riesz decomposition theorem.

2.1 Definitions and known facts

If $\Omega \subseteq \mathbb{C}$ is a domain, then every positive function $\lambda : \Omega \to (0, +\infty)$ on Ω induces a conformal metric, denoted by the linear form $\lambda(z)|dz|$ (see [11, 17]) in our study. In the domain Ω with the metric $\lambda(\omega)|d\omega|$, the distance between two points ω_1 and ω_2 in Ω is defined by

$$\lambda(\omega_1, \omega_2) = \inf \int_\gamma \lambda(\omega)|d\omega|,$$

where γ is a path joining ω_1 and ω_2 in Ω and the infimum is over all possible γ. We call $\lambda(z)$ the density function and $\lambda(\omega_1, \omega_2)$ the distance function of $\lambda(z)|dz|$. If Ω is Euclidean, the density $\lambda \equiv 1$, and $\lambda(A, B) = |A - B|$. The hyperbolic metric on the unit disk $\mathbb{D}$ with the Gaussian curvature -4 is given by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1 - |z|^2}. \tag{2.1.1}$$

If $\mathbb{C}\backslash\Omega$ contains at least two points, Ω is a hyperbolic plane domain, and there exists a universal covering projection $\pi : \mathbb{D} \to \Omega$ by the uniformisation theorem. The hyperbolic density $\lambda_\Omega(\omega)$ on Ω is defined by

$$\lambda_\Omega(\omega) = \frac{\lambda_{\mathbb{D}}(z)}{|\pi'(z)|}, \tag{2.1.2}$$

where $\pi(z) = \omega$. Now we give an infinitesimal property of the hyperbolic metric $\lambda_\Omega(\omega)|d\omega|$ (see [16, Chapter 7]).

Theorem 2.1.1 ([16])**.** *Let $\lambda_\Omega(\omega)|d\omega|$ be the hyperbolic metric on Ω. Then for $\omega \in \Omega$,*

$$\lambda(\omega, \omega + t) = |t|\lambda_\Omega(\omega) + o(t), \qquad \text{as } t \to 0.$$

Since the Euclidean metric $\delta_\Omega(\omega)|d\omega|$ satisfies the formula $\delta_\Omega(\omega, \omega + t) = |t|\delta_\Omega(\omega)$ with $\delta_\Omega \equiv 1$, we have the following corollary of Theorem 2.1.1.

Corollary 2.1.2 ([16])**.** *The hyperbolic metric on any hyperbolic domain is locally equivalent to the Euclidean metric.*

The property stated in Corollary 2.1.2 serves as a basis for an equivalent definition of a conformal metric. The metric $\lambda(z)|dz|$ is *conformal with respect to the Euclidean metric*, or simply *a conformal metric* if the distance function satisfies

$$\lim_{|t|\to 0} \frac{\lambda(z, z+t)}{|t|} = \lambda(z).$$

This infinitesimal property of the conformal metric $\lambda(z)|dz|$ shows that, a domain equipped with $\lambda(z)|dz|$ is locally flat.

For a point $p \in \Omega$, let z be local coordinates such that $z(p) = 0$. We say a conformal metric $\lambda(z)|dz|$ on the punctured domain $\Omega^* := \Omega\backslash\{p\}$ has a *singularity of order $\alpha \leq 1$ at the point p*, if, in local coordinates z,

$$\log \lambda(z) = \begin{cases} -\alpha \log|z| + O(1) & \text{if } \ \alpha < 1 \\ -\log|z| - \log\log(1/|z|) + O(1) & \text{if } \ \alpha = 1, \end{cases} \tag{2.1.3}$$

as $z(p) \to 0$ with O and o being the Landau symbols. We call the point p a *conical singularity or corner of order α* if $\alpha < 1$ and a *cusp* if $\alpha = 1$.

The generalized Gaussian curvature $\kappa_\lambda(z)$ of $\lambda(z)|dz|$ is defined by

$$\kappa_\lambda(z) = -\frac{1}{\lambda(z)^2} \liminf_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} \log\lambda(z + re^{it})dt - \log\lambda(z)\right). \tag{2.1.4}$$

If $\lambda(z)|dz|$ is a regular conformal metric, i.e. the function $\lambda(z)$ is positive and twice continuously differentiable, denoted by $\lambda \in C^2$, then

$$\kappa_\lambda(z) = -\frac{\Delta \log\lambda(z)}{\lambda(z)^2}, \tag{2.1.5}$$

where Δ denotes the Laplace operator (see [34]). It is well known that, if $a < \kappa_\lambda(z) < b < 0$ with constants $a, b \in \mathbb{R}$, the metric $\lambda(z)|dz|$ only has corners or cusps at isolated singularities (see [23]). Usually, we let $u(z) := \log\lambda(z)$. Since we consider an isolated singularity p, there exists a simply connected neighborhood U of p in Ω, so we may set $\Omega = \mathbb{D}$ and $p = 0$. If

$\kappa_\lambda \equiv 0$ in $\mathbb{D}$, λ is the Euclidean metric. From (2.1.5) u satisfies the Laplace equation $\Delta u = 0$, which means u is harmonic on $\mathbb{D}$. If κ_λ is a positive constant in $\mathbb{D}$, $\lambda(z)|dz|$ is the spherical metric (see [29, 1.1, 2.1]). If $\kappa_\lambda(z)$ is a negative constant in $\mathbb{D}$, the corresponding $\lambda(z)|dz|$ is the hyperbolic metric on $\mathbb{D}$ and $\mathbb{D}$ is a model of the hyperbolic space.

A very basic property of the Gaussian curvature $\kappa(z)$ is that $\kappa(z)$ is conformal invariant. If $\lambda(z)|dz|$ is a conformal metric on a plane domain D and $f : \Omega \to D$ is a holomorphic mapping of a Riemann surface Ω into D, then the pullback of $\lambda(z)|dz|$ is defined by

$$(f^*\lambda)(\omega)|d\omega| := \lambda(f(\omega))|f'(\omega)||d\omega|. \tag{2.1.6}$$

It is evident that $f^*\lambda(\omega)|d\omega|$ is a conformal metric on $\Omega \backslash \{\text{critical points of } f\}$ with the Gaussian curvature

$$\kappa_{f^*\lambda}(\omega) = \kappa_\lambda(f(\omega)).$$

In this case, $\lambda(z)|dz|$ is called the pushforward of $\lambda(\omega)|d\omega|$. This is the reason we can obtain (2.1.2) on any Ω from (2.1.1) on $\mathbb{D}$. Using this conformal invariance, we can only consider one Riemann surface Ω with the conformal metric all over the conformal equivalence class of Ω.

The following result shows the equivalence between definitions (2.1.4) and (2.1.5). Here we present it in detail.

Lemma 2.1.3 ([35])**.** *Suppose that $\Omega \subseteq \mathbb{C}$ is a domain. If the function u is of class $C^2(\Omega)$, then for $z \in \Omega$, we have*

$$\lim_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})dt - u(z)\right) = \Delta u(z).$$

Proof. Since $u \in C^2$, fixing $z_0 \in \Omega$, the Taylor's expansion of $u(z)$ at z_0 is

$$\begin{aligned} & u(z_0+z) \\ = \quad & u(z_0) + x\cdot u_x(z_0) + y\cdot u_y(z_0) + \frac{x^2}{2}u_{xx}(z_0) + \frac{y^2}{2}u_{yy}(z_0) + xy\cdot u_{xy}(z_0) + \varepsilon(z), \end{aligned}$$

where $\varepsilon(z) = o(|z|^2)$ as $z \to 0$, $z = x + yi$ (see e.g. [4]). We notice that $x = r\cos t$, $y = r\sin t$, $0 < t \le 2\pi$, and z_0 is fixed, then

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{it})dt - u(z_0) = \frac{r^2}{4}\left(u_{xx}(z_0)+u_{yy}(z_0)\right) + \frac{1}{2\pi}\int_0^{2\pi}\varepsilon(re^{it})dt.$$

When $r = |z| \to 0$,

$$\frac{1}{r^2}\int_0^{2\pi}\varepsilon(re^{it})dt = \int_0^{2\pi}\frac{\varepsilon(re^{it})}{r^2}dt = o(1),$$

thus

$$\lim_{r\to 0}\frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{it})dt - u(z)\right) = u_{xx}(z_0)+u_{yy}(z_0) = \Delta u(z_0)$$

as required. □

We denote $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$. The hyperbolic metric on $\mathbb{D}^*$ with the singularity at the origin of order $\alpha \le 1$ is well known. First we consider the function $f_1(z) = z^{\frac{1}{1-\alpha}}$, $0 < \alpha < 1$, f_1

is conformal on the punctured unit disk $\mathbb{D}^* := \mathbb{D}\backslash\{0\}$. Let $\omega := f_1(z)$ and $\lambda_\alpha(\omega)|d\omega|$ be the hyperbolic metric on $\mathbb{D}^*$. The pull back of $\lambda_\alpha(\omega)|d\omega|$ by f_1, defined by $f_1^*(\lambda_\alpha(\omega)|d\omega|) := \lambda_\alpha(f_1(z))|f_1'(z)||dz|$, should be the hyperbolic metric on $\mathbb{D}$. Combining with (2.1.1) results in

$$\lambda_\alpha(\omega)|d\omega| = \frac{\lambda_{\mathbb{D}}(z)}{|f_1'(z)|}|d\omega| = \frac{|d\omega|}{(1-|z|^2)|f_1'(z)|} = \frac{(1-\alpha)|\omega|^{-\alpha}|d\omega|}{1-|\omega|^{2(1-\alpha)}}. \tag{2.1.7}$$

By the definition (2.1.3) we know the origin is a corner of order $\alpha \in (0,1)$. From the form of f_1 we see that the Riemann surface of $f_1(\mathbb{D}^*)$ looks like an ice-cream cone near $z = 0$, that is why we call the origin a "conical" singularity. It is also customary to say that a conformal metric with a conical singularity 0 of order $0 < \alpha < 1$ has the angle $2\pi(1-\alpha)$ at the origin (see [21]). The shape of the metric function $\lambda_\alpha(z)$ is shown in Figure 2.1 with $\alpha = 0.5$. When $\alpha = 1$, to get the hyperbolic metric $\lambda_1(\omega)|d\omega|$ on $\mathbb{D}^*$ we consider $f_2(z) = \exp\frac{z+1}{z-1}$ from $\mathbb{D}$ onto $\mathbb{D}^*$. Let $\omega := f_2(z)$. Then

$$\lambda_1(\omega)|d\omega| = \frac{|d\omega|}{(1-|z|^2)|f_2'(z)|} = \frac{|d\omega|}{2|\omega|\log(1/|\omega|)}. \tag{2.1.8}$$

Ahlfors [3] mentioned that the differential element $d\omega$ does not depend on the direction of

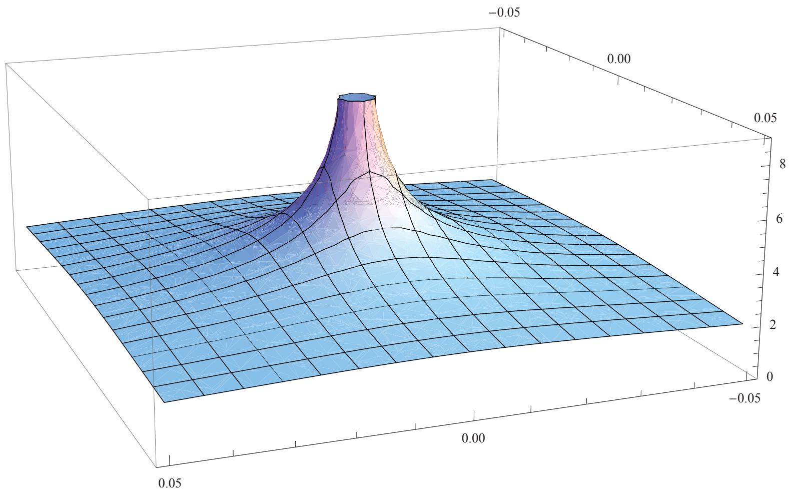

Fig 2.1. Graph $\lambda_{0.5}(z)$ near the origin

dz in (2.1.7) or (2.1.8). For $R > 0$, let $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ and $\mathbb{D}_R^* := \mathbb{D}_R\backslash\{0\}$. The hyperbolic metrics (2.1.7) and (2.1.8) on $\mathbb{D}^*$ can be generalized to the punctured disk $\mathbb{D}_R^*$ by a scalar map $\omega \mapsto R\omega$.

Theorem A ([21]). *For $R > 0$, the hyperbolic density on the punctured disk $\mathbb{D}_R^*$ is defined by*

$$\lambda_{\alpha,R}(z) = \frac{(1-\alpha)R^{1-\alpha}|z|^{-\alpha}}{R^{2(1-\alpha)}-|z|^{2(1-\alpha)}} = \frac{1-\alpha}{2|z|\sinh\left((1-\alpha)\log(R/|z|)\right)}, \quad \alpha < 1, \tag{2.1.9}$$

$$\lambda_{1,R}(z) = \frac{1}{2|z|\log(R/|z|)} \tag{2.1.10}$$

for $z \in \mathbb{D}_R^*$.

The concept of SK-metrics was given by Heins in [11], but its initial idea came from Ahlfors in [3] and [4]. Heins used definition (2.1.4) to define SK-metrics. A metric is SK-metric, if it is conformal and has the Gaussian curvature bounded above by -4. According to Heins, the investigation is strongly motivated by the analogy between SK-metrics and subharmonic functions, since, from (2.1.5) we know that, if $\lambda(z)|dz|$ is an SK-metric on a domain Ω, $u(z)$ is a subharmonic function on Ω. These striking similarities of regular conformal metric of curvature -4 and SK-metric with harmonic and subharmonic functions can be also found in [28]. For SK-metrics, there is a generalization of the maximum principle mentioned by Ahlfors [3, Theorem A] and Heins [11, Theorem 2.1], which claims that the hyperbolic metric is the unique maximal SK-metric on $\mathbb{D}$.

Theorem 2.1.4 ([3]). (Ahlfors′ lemma) *Let $d\sigma$ be the hyperbolic metric on $\mathbb{D}$ given in (2.1.1) and ds be the metrics on $\mathbb{D}$ induce by an SK-metric on some Riemann surface W. If the function $f(z)$ is analytic in $\mathbb{D}$, then the inequality*

$$ds \leq d\sigma$$

will hold throughout the circle.

The following result is a variant of Theorem 2.1.4.

Theorem 2.1.5 ([11]). *Suppose that W is a relatively compact domain of Ω and that $\lambda(w)$ is an SK-metric on W, $\mu(w)$ is a pullback on W of $\lambda_{\mathbb{D}}(z)$ defined in (2.1.1). If for all $p \in \partial W$,*

$$\limsup_{w \to p} \frac{\lambda(w)}{\mu(w)} \leq 1,$$

then throughout the boundary ∂W, it holds that

$$\lambda(w) \leq \mu(w).$$

In the punctured disk $\mathbb{D}_R^*$, Kraus, Roth and Sugawa gave the expressions (2.1.9) and (2.1.10) of the hyperbolic metric with a singularity at the origin of order $\alpha \leq 1$ in [21] without any detailed discussion. Now we give a complete presentation of the proof as follows.

Theorem 2.1.6 ([21, 35]). *Let $\lambda_{\alpha,R}(z)$ be given by Theorem A. Then for an arbitrary SK-metric $\sigma(z)|dz|$ on $\mathbb{D}_R^*$ with a singularity at $z = 0$ of order $\alpha \leq 1$, we have $\sigma(z) \leq \lambda_{\alpha,R}(z)$ on $\mathbb{D}_R^*$.*

Proof. When $\alpha < 1$, we choose an arbitrary R_0, $0 < R_0 < R$, and consider

$$\lambda_{\beta,R_0}(z) = \frac{(1-\alpha)R_0^{1-\alpha}|z|^{-\alpha}}{R_0^{2(1-\alpha)} - |z|^{2(1-\alpha)}}$$

on $0 < |z| < R_0$ for $\alpha < \beta < 1$. Let $u(z) := \log \sigma(z)$, $v(z) := \log \lambda_{\beta,R_0}(z)$, $E := \{z : 0 < |z| < R_0, u(z) > v(z)\}$.

Now we have the assertion that $0 \notin \overline{E}$. Since $\sigma(z)|dz|$ and $\lambda_{\beta,R_0}(z)|dz|$ both have the singularity at $z = 0$ with order α, β, respectively, then

$$v(z) = -\beta \log|z| + O(1), \ u(z) = -\alpha \log|z| + O(1),$$

and $u - v = (\beta - \alpha)\log|z| + O(1)$. Since $u - v \to -\infty$ as $z \to 0$, on a sufficiently small neighborhood of $z = 0$, $u - v < 0$ holds, thus $0 \notin \overline{E}$.

Similarly, we have $\partial E \cap \{z : |z| = R_0\} = \emptyset$, because $v \to +\infty$ as $|z| \to R_0$, and u is bounded in $\{z : |z| = R_0\}$. Hence $\partial E \subseteq \mathbb{D}^*_{R_0}$. This means that, if $z_0 \in \partial E$, $z_0 \in \mathbb{D}^*_{R)}$ and $u(z_0) - v(z_0) = 0$.

Functions $v(z)$ and $u(z)$ satisfy

$$\lim_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} v(z + re^{it})dt - v(z)\right) = e^{2v(z)},$$

and

$$\liminf_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})dt - u(z)\right) \geq e^{2u(z)}$$

by Lemma 2.1.3. Then

$$\liminf_{r\to 0} \frac{4}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi} \left(u(z + re^{it}) - v(z + re^{it})\right)dt - (u(z) - v(z))\right) \geq e^{2u(z)} - e^{2v(z)},$$

where $e^{2u} - e^{2v}$ is positive on E. Thus for $z \in E$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} \left(u(z + re^{it}) - v(z + re^{it})\right)dt - (u(z) - v(z)) \geq 0,$$

that is,

$$u(z) - v(z) \leq \frac{1}{2\pi}\int_0^{2\pi} \left(u(z + re^{it}) - v(z + re^{it})\right)dt.$$

Now we recall the definition of *subharmonic* functions. Let Ω be an open subset of $\mathbb{C}$. A function $u : \Omega \to [-\infty, \infty)$ is called subharmonic if u is upper semi-continuous and satisfies the local sub-mean inequality, i.e. given $z \in \Omega$, there exists $\rho > 0$ such that for all positive r, $0 \leq r < \rho$,

$$u(z) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})dt \tag{2.1.11}$$

holds. If we adopt definition (2.1.11), $u - v$ is subharmonic on E, hence $u - v$ has no maximum in E and $u - v$ approaches the least upper bound on a sequence tending to ∂E. This contradicts the fact that $u = v$ on ∂E. Thus $E = \emptyset$.

Letting $R_0 \to R$ and $\beta \to \alpha$ gives the maximality of $\lambda_{\alpha,R}(z)$ for $\alpha < 1$. According to Kraus, Roth and Sugawa [21], when $\alpha = 1$ the expression (2.1.9) is in the limit sense $\lim_{\alpha\to 1}$ to obtain $\lambda_{1,R}(z)$ in (2.1.10). This completes the proof. □

Remark 2.1.7. *The righthand side of (2.1.11) is called the circumferential mean of function u. Heins used it to describe the curvature in the definition of SK-metrics in [11] with $\rho = 1$ and $z = 0$.*

2.2 Hypergeometric functions and Gamma functions

For complex numbers a, b, c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined by

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\qquad |z|<1,$$

where $(a)_n$ is the Pochhammer symbol, namely, $(a)_0=1$ and

$$(a)_n=a(a+1)\cdots(a+n-1)$$

for $n=1,2,3,\ldots$. It is continued analytically to the slit plane $\mathbb{C}\backslash[1,+\infty)$. The derivative of $F(a,b,c;z)$ is given by

$$\frac{d}{dz}F(a,b,c;z)=\frac{ab}{c}F(a+1,b+1,c+1;z). \tag{2.2.12}$$

We can immediately obtain

$$\frac{d^n}{dz^n}F(a,b,c;z)=\frac{(a)_n(b)_n}{(c)_n}F(a+n,b+n,c+n;z). \tag{2.2.13}$$

We have

$$\begin{aligned} F(a,b,c;z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)\\ &+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c-a-b+1;1-z) \end{aligned} \tag{2.2.14}$$

for $|\arg(1-z)|<\pi$, where $\Gamma(z)$ is Gamma function (see 15.3.6 in [1]). Each term of (2.2.14) has a pole when $c=a+b\pm n$, $n=0,\ 1,\ 2,\ \ldots$, and this case is covered by

$$\begin{aligned} &F(a,b,a+b+n;z)\\ =&\frac{\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+n)\Gamma(b+n)}\sum_{j=0}^{n-1}\frac{(a)_j(b)_j}{j!(1-n)_j}(1-z)^j\\ &-\frac{\Gamma(a+b+n)}{\Gamma(a)\Gamma(b)}(z-1)^n\sum_{j=0}^{\infty}\frac{(a+n)_j(b+n)_j}{j!(j+n)!}(1-z)^j[\log(1-z)\\ &-\Psi(j+1)-\Psi(j+n+1)+\Psi(a+j+n)+\Psi(b+j+n)], \end{aligned} \tag{2.2.15}$$

for $|1-z|<1$ and $|\arg(1-z)|<\pi$, where $\Psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function (see 15.3.11 in [1]). We take the convention that $\sum_{j=a}^{b}=0$ if $b<a$ through our study. The behavior of $F(a,b,c;z)$ near $z=1$ satisfies

$$\begin{cases} F(a,b,c;1)=\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \text{if } a+b<c,\\ F(a,b,a+b;z)=\dfrac{1}{B(a,b)}\left(\log\dfrac{1}{1-z}+R(a,b)\right)(1+O(1-z)),\\ F(a,b,c;z)=(1-z)^{c-a-b}F(c-a,c-b,c;z),\ \text{if } a+b>c. \end{cases} \tag{2.2.16}$$

Here

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2.2.17}$$

is the beta function and

$$R(a,b) = 2\Psi(1) - \Psi(a) - \Psi(b). \tag{2.2.18}$$

The asymptotic formula in (2.2.16) for the case $a+b=c$ is due to Ramanujan (see [1, 6]). Let $G(x) := 2\left(\Psi(1) - \Psi(x)\right) - \pi \cot \pi x$. For the Gamma function $\Gamma(z)$ and $0 < x < 1$, we have $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$. Taking the logarithmic derivatives of both sides leads to

$$\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1-x)}{\Gamma(1-x)} = -\pi \cot \pi x.$$

Thus

$$G(x) = 2\Psi(1) - \Psi(x) - \Psi(1-x), \tag{2.2.19}$$

which means $G(x) = G(1-x)$. The fact that the digamma function is negative and decreasing on (0, 1) implies that $G(x) > 0$ when $0 < x < 1$. The graph of $G(x)$ on $(0,1)$ is shown in Figure 2.2, where $G(0.5) = 2.7725\ldots$.

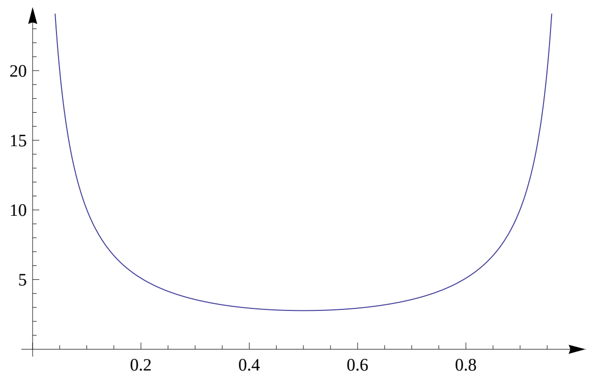

Fig 2.2. Function $G(x)$

We will need the higher order derivatives of $\log\log(1/|z|)$. By induction we know

$$\partial^n \log\log(1/|z|) = \sum_{j=1}^{n} \frac{C_j^{(n)}}{z^n \log^j(1/|z|)} \tag{2.2.20}$$

with constant $C_j^{(n)}$ for $1 \le j \le n$. As for the pure derivative $\partial^n \log\log(1/|z|)$ with $n \ge 1$, set $\mathcal{A}_n := C_1^{(n)}$ and $\mathcal{B}_n := C_2^{(n)}$, then

$$\partial^n \log\log(1/|z|) = \frac{\mathcal{A}_n}{z^n \log(1/|z|)} + \frac{\mathcal{B}_n}{z^n \log^2(1/|z|)} + \sum_{j=3}^{n} \frac{C_j^{(n)}}{z^n \log^j(1/|z|)},$$

thus the following recurrent relations hold,

$$\mathcal{A}_1 = -\frac{1}{2}, \quad \mathcal{B}_1 = 0,$$

$$\mathcal{A}_n = -(n-1)\mathcal{A}_{n-1}, \quad \mathcal{B}_n = -(n-1)\mathcal{B}_{n-1} + \frac{1}{2}\mathcal{A}_{n-1}.$$

Therefore

$$\mathcal{A}_n = \frac{(-1)^n}{2}(n-1)!, \tag{2.2.21}$$

$$\mathcal{B}_n = \frac{(-1)^{n-1}}{4}(n-1)!\sum_{j=1}^{n-1}\frac{1}{j}. \tag{2.2.22}$$

For the mixed derivative case with $n \geq 1$, $m \geq 1$, we fix n, by induction,

$$\bar{\partial}^m \partial^n \log\log(1/|z|) = \sum_{j=1}^{m} \frac{C_j^{(m,\,n)}}{\bar{z}^m z^n \log^{j+1}(1/|z|)} \tag{2.2.23}$$

with constant $C_j^{(m,\,n)}$ for $1 \leq j \leq m$. Set $\mathcal{C}_m := C_1^{(m,\,n)}$ and $\mathcal{D}_m := C_2^{(m,\,n)}$, we have

$$\begin{aligned} & \bar{\partial}^m \partial^n \log\log(1/|z|) \\ = \quad & \frac{\mathcal{C}_m}{\bar{z}^m z^n \log^2(1/|z|)} + \frac{\mathcal{D}_m}{\bar{z}^m z^n \log^3(1/|z|)} + \sum_{j=3}^{m} \frac{C_j^{(m,\,n)}}{\bar{z}^m z^n \log^{j+1}(1/|z|)}. \end{aligned} \tag{2.2.24}$$

Then

$$\mathcal{C}_1 = \frac{1}{2}\mathcal{A}_n, \quad \mathcal{D}_1 = \mathcal{B}_n,$$

$$\mathcal{C}_m = -(m-1)\mathcal{C}_{m-1}, \quad \mathcal{D}_m = -(m-1)\mathcal{D}_{m-1} + \mathcal{C}_{m-1}.$$

Therefore

$$\mathcal{C}_m = \frac{(-1)^{m+n-1}}{4}(m-1)!(n-1)!, \tag{2.2.25}$$

$$\mathcal{D}_m = \frac{(-1)^{m+n}}{4}(m-1)!(n-1)!\left(\sum_{j=1}^{n-1}\frac{1}{j} + \sum_{j=1}^{m-1}\frac{1}{j}\right). \tag{2.2.26}$$

2.3 Logarithmic potentials

Since we can employ a way related to partial differential equations to investigate the asymptotic behavior of u near the origin, potential theory is a powerful tool in our case. This section only refers to the logarithmic potential. We present a formula of the higher order derivatives for the logarithmic potential.

For a locally bounded, integrable function $f(z)$ defined on a domain $\Omega \subseteq \mathbb{C}$, the integral

$$\frac{1}{2\pi}\int_{\Omega} L(z-\zeta) f(\zeta) d\sigma_\zeta$$

is called the logarithmic potential of f, where

$$L(z-\zeta) = \log|z-\zeta|, \tag{2.3.27}$$

and $d\sigma_\zeta$ is the area element on the domain Ω. The Hölder spaces $C^{n,\nu}(\mathbb{D}_R)$ are defined as the subspaces of $C^n(\mathbb{D}_R)$ consisting of functions whose $n-$th order partial derivatives are locally Hölder continuous with exponent ν in $\mathbb{D}_R$, $0 < \nu \leq 1$. We identify $\mathbb{C}$ with $\mathbb{R}^2$, and set

$0 < r \le 1$. Write $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$, and for $n \ge 1$ denote

$$\frac{\partial^n}{\partial x_1^n} = \partial_1^n, \; \frac{\partial^n}{\partial x_2^n} = \partial_2^n. \tag{2.3.28}$$

We will need the divergence theorem. For a point $z = (x_1, x_2)$, a vector field $\mathbf{w}(z) = (w_1(z), w_2(z))$ and a function $u(z)$, denote

$$\operatorname{div} \mathbf{w} = \partial_1 w_1 + \partial_2 w_2 = \text{divergence of } \mathbf{w},$$

$$Du = (\partial_1 u, \partial_2 u) = \text{gradient of } u,$$

then $\Delta u = \operatorname{div} Du$ with Δ being the Laplace operator.

Theorem 2.3.1 (e.g. [10])**.** (Divergence Theorem) *Let Ω be a bounded domain with C^1 boundary $\partial\Omega$. Then for any vector field $\mathbf{w}$ in $C^0(\bar{\Omega}) \cap C^1(\Omega)$, we have*

$$\iint_\Omega \operatorname{div} \mathbf{w} d\sigma_z = \int_{\partial\Omega} \langle N(z), \mathbf{w}\rangle \, |dz|, \tag{2.3.29}$$

where $\langle \; , \; \rangle$ is the inner product, $N(z) = \langle N_1(z), N_2(z)\rangle$ is the unit outward normal at $z \in \partial\Omega$.

We let $\mathbf{w}(z) = v(z)\, Du(z)$. Then $\operatorname{div} \mathbf{w} = Du\, Dv + v\, \Delta u$, and (2.3.29) becomes

$$\iint_\Omega Du\, Dv\, d\sigma_z + \iint_\Omega v\, \Delta u\, d\sigma_z = \int_{\partial\Omega} v\, \langle Du, N(z)\rangle\, |dz|. \tag{2.3.30}$$

In (2.3.30), select one variable x_m for $m = 1, 2$, fix the other component x_{3-m} and denote $u_m := \partial_m u$, we obtain *Green's first identity*,

$$\iint_\Omega u_m\, \partial_m v\, d\sigma_z + \iint_\Omega v\, \partial_m u_m\, d\sigma_z = \int_{\partial\Omega} v\, u_m\, N_m(z)\, |dz|. \tag{2.3.31}$$

The following proposition is for the first and the second order derivatives of the logarithmic potential.

Proposition 2.3.2 ([10, 19])**.** *Let $f : \mathbb{D}_r \to \mathbb{R}$ be a locally bounded, integrable function in $\mathbb{D}_r$ and ω be the logarithmic potential of f. Then $\omega \in C^1(\mathbb{D}_r)$ and for any $z = x_1 + ix_2 \in \mathbb{D}_r$, $j \in \{1, 2\}$,*

$$\partial_j \omega(z) = \frac{1}{2\pi} \iint_{\mathbb{D}_r} \partial_j L(z - \zeta) \cdot f(\zeta) d\sigma_\zeta.$$

If, in addition, f is locally Hölder continuous with exponent $\nu \le 1$, then $\omega \in C^2(\mathbb{D}_r)$, $\Delta\omega = f$ on $\mathbb{D}_r$ and for $z \in \mathbb{D}_r$,

$$\begin{aligned} & \partial_l \partial_j \omega(z) \\ = \; & \frac{1}{2\pi} \iint_{\mathbb{D}_R} \partial_l \partial_j L(z - \zeta)\, (f(\zeta) - f(z))\, d\sigma_\zeta - \frac{1}{2\pi} f(z) \int_{\partial\mathbb{D}_R} \partial_j L(z - \zeta) N_l(\zeta) |d\zeta|, \end{aligned}$$

where $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ is the unit outward normal at the point $\zeta \in \partial\mathbb{D}_R$, $R > r$ such that Theorem 2.3.1 holds on $\mathbb{D}_R$ and f is extended to vanish outside $\mathbb{D}_r$.

There is a similar proposition for higher order derivatives of the logarithmic potential. To state it we need the notation of the multi-index. Define a multi-index $\boldsymbol{j} = (j_1, j_2)$, $|\boldsymbol{j}| = j_1 + j_2$,

$j_1, j_2 = 0, 1, 2, \dots$, such that $(\zeta - z)^{\boldsymbol{j}} = (y_1 - x_1)^{j_1}(y_2 - x_2)^{j_2}$, $\boldsymbol{j}! = j_1! j_2!$. For $z = x_1 + ix_2$, denote $\partial^{\boldsymbol{j}} = \partial_1^{j_1}\partial_2^{j_2}$, where ∂_1, ∂_2 are given in (2.3.28). For a given multi-index $\boldsymbol{j} = (j_1, j_2)$, we can choose $\boldsymbol{e}_\tau = (0,1)$ or $(1,0)$ for $\tau = 1, 2, \dots$ such that $\boldsymbol{j} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \cdots + \boldsymbol{e}_n$ with $n = |\boldsymbol{j}|$. For a function $f \in C^{|\boldsymbol{j}|}$, define $P_n[f]$ by

$$P_n[f](z,\zeta) := \begin{cases} \displaystyle\sum_{|\boldsymbol{a}| \le n} \frac{(\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}}}{\boldsymbol{a}!} f(z) & \text{if } \ n \ge 1 \\ f(z) & \text{if } \ n = 0, \end{cases}$$

where $\boldsymbol{a}$ is a multi-index. We have the following recurrent formula for $P_n[f](z,\zeta)$.

Lemma 2.3.3. *For $P_n[f](z,\zeta)$ defined as above, we have*

$$\frac{\partial^{\boldsymbol{e}}}{\partial \zeta} P_n[f](z,\zeta) = P_{n-1}[\partial^{\boldsymbol{e}} f](z,\zeta)$$

hold for $\boldsymbol{e} = (0,1)$ or $(1,0)$.

Proof. We take the case $\boldsymbol{e} = (1,0)$ as an example, when $\boldsymbol{e} = (0,1)$ it is similar. Let $n \ge 1$. Then,

$$\begin{aligned} \frac{\partial}{\partial y_1} P_n[f](z,\zeta) &= \frac{\partial}{\partial y_1} \sum_{\substack{\boldsymbol{a}_1 + \boldsymbol{a}_2 \le n \\ 0 \le \boldsymbol{a}_1 \le n}} \frac{(y_1 - x_1)^{\boldsymbol{a}_1}(y_2 - x_2)^{\boldsymbol{a}_2}}{\boldsymbol{a}_1! \boldsymbol{a}_2!} {\partial_1}^{\boldsymbol{a}_1} {\partial_2}^{\boldsymbol{a}_2} f(z) \\ &= \sum_{\substack{\boldsymbol{a}_1 + \boldsymbol{a}_2 \le n \\ 1 \le \boldsymbol{a}_1 \le n}} \frac{(y_1 - x_1)^{\boldsymbol{a}_1 - 1}(y_2 - x_2)^{\boldsymbol{a}_2}}{(\boldsymbol{a}_1 - 1)! \boldsymbol{a}_2!} {\partial_1}^{\boldsymbol{a}_1} {\partial_2}^{\boldsymbol{a}_2} f(z) \\ &= \sum_{\substack{(\boldsymbol{a}_1 - 1) + \boldsymbol{a}_2 \le n-1 \\ 0 \le \boldsymbol{a}_1 - 1 \le n}} \frac{(y_1 - x_1)^{\boldsymbol{a}_1 - 1}(y_2 - x_2)^{\boldsymbol{a}_2}}{(\boldsymbol{a}_1 - 1)! \boldsymbol{a}_2!} {\partial_1}^{\boldsymbol{a}_1 - 1} {\partial_2}^{\boldsymbol{a}_2} \, \partial_1 f(z) \\ &= \sum_{\boldsymbol{a}_1 + \boldsymbol{a}_2 \le n-1} \frac{(y_1 - x_1)^{\boldsymbol{a}_1}(y_2 - x_2)^{\boldsymbol{a}_2} {\partial_1}^{\boldsymbol{a}_1} {\partial_2}^{\boldsymbol{a}_2}}{\boldsymbol{a}_1! \boldsymbol{a}_2!} \partial_1 f(z) \\ &= \sum_{|\boldsymbol{a}| \le n-1} \frac{(\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}}}{\boldsymbol{a}!} \partial_1 f(z) = P_{n-1}[\partial_1 f](z,\zeta). \end{aligned}$$

□

We can use the multi-index notation to rewrite the Taylor expansion of f.

Theorem 2.3.4 (e.g. [4])**.** *If $f(\zeta)$ is analytic in a domain $\Omega \subseteq \mathbb{C}$, near a fixed point $z \in \Omega$, it is possible to write*

$$f(\zeta) = \sum_{t=0}^{n} \frac{f^{(t)}(z)}{t!}(\zeta - z) + R_{n+1}(z,\zeta),$$

where $R_{n+1}(z,\zeta)$ is the error term and $R_{n+1}(z,\zeta) = f_{n+1}(z)(\zeta - z)^{n+1}$ with $f_{n+1}(z)$ analytic in Ω. This expression is equivalent to

$$f(\zeta) = P_n[f](z,\zeta) + R_{n+1}(z,\zeta), \tag{2.3.32}$$

with $R_{n+1}(z,\zeta)$ as above.

Remark 2.3.5. *If $f \in C^{n,\nu}(\Omega)$ with $0 < \nu \le 1$, $n \ge 1$, note that only the local Hölder continuity is considered here, the error term $R_{n+1}(z,\zeta)$ satisfies*

$$R_{n+1}(z,\zeta) = O(|z-\zeta|^{\nu+n}). \tag{2.3.33}$$

On the basis of Lemma 2.3.3, we can present the analogue of Proposition 2.3.2 as follows.

Proposition 2.3.6 ([35])**.** *Suppose that $0 < r < R$, $f : \mathbb{D}_r \to \mathbb{R}$ is a locally bounded, integrable function, and $f \in C^{n-2,\nu}(\mathbb{D}_r)$ with $0 < \nu \le 1$. Let $n \ge 2$, ω be the logarithmic potential of f. Then $\omega(z) \in C^n(\mathbb{D}_r)$ and for a multi-index $\boldsymbol{j}$, $\boldsymbol{j} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n$, $|\boldsymbol{j}| = n$,*

$$\begin{aligned} &\partial^{\boldsymbol{j}}\omega(z) \\ = &\frac{1}{2\pi}\int_{\mathbb{D}_R} \partial^{\boldsymbol{j}} L(z-\zeta)\cdot(f(\zeta) - P_{n-2}[f](z,\zeta))\,d\sigma_\zeta \\ &-\frac{1}{2\pi}\sum_{\tau=1}^{n-1}\int_{\partial\mathbb{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](z,\zeta)\cdot\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle|d\zeta| \end{aligned} \tag{2.3.34}$$

for $z \in \mathbb{D}_r$, where $\boldsymbol{\theta}_\tau := \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_\tau$, $\boldsymbol{\phi}_\tau := \boldsymbol{e}_{\tau+2} + \cdots + \boldsymbol{e}_n$ for $\tau = 1, \dots, n-2$ and $\boldsymbol{\phi}_{n-1} := (0,0)$, $N(\zeta) = (N_1(\zeta), N_2(\zeta))$ is the unit outward normal at the point $\zeta \in \partial\mathbb{D}_R$ with $R > r$ such that Theorem 2.3.1 holds on $\mathbb{D}_R$, $\langle\ ,\ \rangle$ is the inner product and the function f is extended to vanish outside $\mathbb{D}_r$.

Proof. Let

$$\begin{aligned} u_{\boldsymbol{j}}(z) \ = \ &\frac{1}{2\pi}\iint_{\mathbb{D}_R} \partial^{\boldsymbol{j}} L(z-\zeta)\cdot(f(\zeta) - P_{n-2}[f](z,\zeta))\,d\sigma_\zeta \\ &-\frac{1}{2\pi}\sum_{\tau=1}^{n-1}\int_{\partial\mathbb{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta)\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](z,\zeta)\cdot\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle|d\zeta|. \end{aligned} \tag{2.3.35}$$

Note that

$$\left|\partial^{\boldsymbol{j}} L(z-\zeta)\right| \le \frac{n!}{|z-\zeta|^n}, \tag{2.3.36}$$

for $n = |\boldsymbol{j}|$, and $L(z-\zeta)$ is harmonic for $\zeta \ne z$. By the local Hölder continuity of f in $\mathbb{D}_r$, the function $u_{\boldsymbol{j}}(z)$ is well defined.

Now we can employ induction. Since Proposition 2.3.2 holds, and $\boldsymbol{j}$ has the decomposition $\boldsymbol{j} = \boldsymbol{\theta}_{n-1} + \boldsymbol{e}_n$, we may assume that the formula (2.3.34) is true for $\boldsymbol{\theta}_{n-1}$. Fix a function $\eta(t) \in C^{n-1}(\mathbb{R})$ such that $0 \le \eta \le 1$, $0 \le \eta' \le 2$, $\eta(t) = 0$ for $t \le 1$, $\eta(t) = 1$ for $t \ge 2$, and set

$$\eta_\varepsilon := \eta\left(\frac{|z-\zeta|}{\varepsilon}\right),\ L := \frac{1}{2\pi}L(z-\zeta).$$

Note that η_ε and L are both skew symmetric with respect to x_1 and y_1, x_2 and y_2. Then by Lemma 2.3.3,

$$\partial^{\boldsymbol{e}} L\eta_\varepsilon = -\frac{\partial^{\boldsymbol{e}}}{\partial\zeta} L\eta_\varepsilon \tag{2.3.37}$$

for $\boldsymbol{e}=(0,1)$ or $(1,0)$. For $\varepsilon>0$, define the function

$$v_{\boldsymbol{j}}(z,\varepsilon):=\iint_{\mathbb{D}_r}\partial^{j}L\eta_\varepsilon\cdot f(\zeta)d\sigma_\zeta.$$

We obtain $v_{\boldsymbol{\theta}_{n-1}}(z,\varepsilon)\in C^{n-1}(\mathbb{D}_r)$ for a fixed ε by induction. From (2.3.27), we know that $\zeta=z$ is a singularity of $L(z-\zeta)$ when $|\boldsymbol{j}|\geq 3$. To overwhelm the blow-up behavior of $L(z-\zeta)$ near the singularity $\zeta=z$, we need the Taylor expansion (2.3.32). To prevent the point $\zeta=z$ from appearing on the boundary $\partial\mathbb{D}_r$, we have to extend the domain $\mathbb{D}_r$ of the integral (2.3.34) into a larger domain $\mathbb{D}_R$ where Theorem 2.3.1 holds. Thus for a sufficiently small ε,

$$\begin{aligned}
&\partial^{\boldsymbol{e}_n}v_{\boldsymbol{j}}(z,\varepsilon)=\iint_{\mathbb{D}_r}\partial^{\boldsymbol{e}_n}(\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon)\cdot f(\zeta)d\sigma_\zeta\\
=\ &\iint_{\mathbb{D}_R}\partial^{\boldsymbol{e}_n}(\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon)\cdot\left(f(\zeta)-P_{n-2}[f]\right)d\sigma_\zeta+\iint_{\mathbb{D}_R}\partial^{\boldsymbol{e}_n}(\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon)\cdot P_{n-2}[f]d\sigma_\zeta.
\end{aligned}$$

Combining the skew symmetry (2.3.37), Green's identity (2.3.31) and Lemma 2.3.3, for the sufficiently small ε, we have

$$\begin{aligned}
&\iint_{\mathbb{D}_R}\partial^{\boldsymbol{e}_n}(\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon)\cdot P_{n-2}[f]d\sigma_\zeta=-\iint_{\mathbb{D}_R}\frac{\partial^{\boldsymbol{e}_n}}{\partial\zeta}\partial^{\boldsymbol{e}_n}(\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon)\cdot P_{n-2}[f]d\sigma_\zeta\\
=\ &-\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_{n-1}}L\cdot P_{n-2}[f]\langle N(\zeta),\boldsymbol{e}_n\rangle|d\zeta|+\iint_{\mathbb{D}_R}\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon\cdot P_{n-3}[\partial^{\boldsymbol{e}_n}f]d\sigma_\zeta\\
=\ &\dots\\
=\ &-\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_{n-1}}L\cdot P_{n-2}[f]\langle N(\zeta),\boldsymbol{e}_n\rangle|d\zeta|-\dots-\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_2}L\cdot P_1[\partial^{\boldsymbol{\phi}_2}f]\langle N(\zeta),\boldsymbol{e}_3\rangle|d\zeta|\\
&+\iint_{\mathbb{D}_R}\partial^{\boldsymbol{\theta}_2}L\eta_\varepsilon\cdot P_0[\partial^{\boldsymbol{\phi}_1}f]d\sigma_\zeta\\
=\ &-\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_{n-1}}L\cdot P_{n-2}[f]\langle N(\zeta),\boldsymbol{e}_n\rangle|d\zeta|-\dots-\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_2}L\cdot P_1[\partial^{\boldsymbol{\phi}_2}f]\langle N(\zeta),\boldsymbol{e}_3\rangle|d\zeta|\\
&-\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_1}L\cdot P_0[\partial^{\boldsymbol{\phi}_1}f]\langle N(\zeta),\boldsymbol{e}_2\rangle|d\zeta|\\
=\ &-\sum_{\tau=1}^{n-1}\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_\tau}L\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau}f]\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle|d\zeta|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\partial^{\boldsymbol{e}_n}v_{\boldsymbol{j}}(z,\varepsilon)\ &=\ \iint_{\mathbb{D}_R}\partial^{\boldsymbol{e}_n}(\partial^{\boldsymbol{\theta}_{n-1}}L\eta_\varepsilon)\cdot\left(f(\zeta)-P_{n-2}[f]\right)d\sigma_\zeta\\
&\quad-\sum_{\tau=1}^{n-1}\int_{\partial\mathbb{D}_R}\partial^{\boldsymbol{\theta}_\tau}L\cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau}f]\langle N(\zeta),\boldsymbol{e}_{\tau+1}\rangle|d\zeta|.
\end{aligned}\tag{2.3.38}$$

Now we compare (2.3.35) and (2.3.38). By the local Hölder continuity of f, Theorem 2.3.4

and the estimate (2.3.33), there exist constants M_1 and M_2 such that

$$\begin{aligned}
|u_{\boldsymbol{j}}(z) - \partial^{\boldsymbol{e}_n} v_{\boldsymbol{j}}(z,\varepsilon)| &= \left| \iint_{|\zeta - z| \leq 2\varepsilon} \left(\partial^{\boldsymbol{j}} L - \partial^{\boldsymbol{j}} L \eta_\varepsilon\right) R_{n-1}(z,\zeta) d\sigma_\zeta \right| \\
\leq \quad M_1 \iint_{|\zeta - z| \leq 2\varepsilon} & \left(\frac{n!}{|\zeta - z|^n} + \frac{2(n-1)!}{\varepsilon |\zeta - z|^{n-1}} \right) |z - \zeta|^{\nu + n - 2} d\sigma_\zeta \\
= \quad M_1 \iint_{|\zeta - z| \leq 2\varepsilon} & \left(\frac{n!}{|\zeta - z|^2} + \frac{2(n-1)!}{\varepsilon |\zeta - z|} \right) |z - \zeta|^{\nu} d\sigma_\zeta \leq M_2 \cdot (2\varepsilon)^\nu
\end{aligned}$$

The last inequality comes from Lemma 4.2 in [10]. Hence $\partial^{\boldsymbol{e}_n} v_{\boldsymbol{j}}(z,\zeta)$ converges to $u_{\boldsymbol{j}}(z)$ uniformly on any compact subset of $\mathbb{D}_r$ as $\varepsilon \to 0$. It is easy to see $v_{\boldsymbol{j}}(z,\varepsilon)$ converges uniformly to $\partial^{\boldsymbol{\theta}_{n-1}} \omega$ in the disk $\mathbb{D}_r$, then $\omega \in C^n(\mathbb{D}_r)$ and $u_{\boldsymbol{j}}(z) = \partial^{\boldsymbol{j}} \omega(z)$. The proof is complete. $\square$

We need the following two results for the future proof. Lemma 2.3.7 is an immediately consequence of the standard regularity theorem (see, e.g. [10, Theorem 6.17]). Lemma B is a corollary of the Riesz decomposition theorem, and can be obtained from Theorem 4.5.1 and Exercise 3.7.3 in [28].

Lemma 2.3.7. (Regularity theorem) *Let u be a C^2-solution to the equation $\Delta u = -\kappa(z) e^{2u}$ in $\mathbb{D}^*$, where $\kappa \in C^{n,\nu}(\mathbb{D}^*)$. Then $u \in C^{n+2,\nu}(\mathbb{D}^*)$. If κ lies in $C^\infty(\mathbb{D}^*)$, then $u \in C^\infty(\mathbb{D}^*)$.*

Lemma B ([21, 28])**.** *Let u be a subharmonic function on $\mathbb{D}_r$ such that $u \in C^2({\mathbb{D}_r}^*)$, Δu is integrable in $\mathbb{D}_r$ and*

$$\lim_{r \to 0} \frac{\sup_{|z|=r} u(z)}{\log(1/r)} = 0.$$

Then $u(z) = h(z) + \omega(z)$ for $z \in \mathbb{D}_r$, where h is a harmonic function on $\mathbb{D}_r$ and $\omega(z)$ is the logarithmic potential of Δu.

Chapter 3

Asymptotic behavior near isolated singularities

In this chapter we investigate the asymptotic behavior near the origin of the higher order derivatives of the remainder functions for a conformal metric. This kind of behavior was described up to the second order derivatives by Kraus and Roth in their 2008 paper [19]. Our work refines their result for the second order mixed derivatives and estimates for higher order derivatives near an isolated singularity.

3.1 Introduction

We denote

$$\partial^n = \frac{\partial^n}{\partial z^n},\ \bar{\partial}^n = \frac{\partial^n}{\partial \bar{z}^n}$$

for $n \geq 1$. Then the following result holds.

Theorem C ([19]). *Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a (locally) Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a C^2-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$, then u has an order $\alpha \in (-\infty, 1]$ at the origin. Define the remainder functions $v(z)$ and $w(z)$ by*

$$\begin{aligned} u(z) &= -\alpha \log|z| + v(z), && \text{if } \alpha < 1, \\ u(z) &= -\log|z| - \log\log(1/|z|) + w(z), && \text{if } \alpha = 1, \end{aligned}$$

according to the value of α. Then $v(z)$ and $w(z)$ are continuous in $\mathbb{D}$. Moreover, concerning the first partial derivatives with respect to z and $\bar{z}$,

$$\partial v(z),\ \bar{\partial} v(z) \text{ are continuous at } z = 0 \qquad \text{if } \alpha < 1/2;$$

and

$$\begin{aligned}
&\partial v(z),\, \bar{\partial} v(z) = O(1) && \text{if } \alpha = 1/2;\\
&\partial v(z),\, \bar{\partial} v(z) = O(|z|^{1-2\alpha}) && \text{if } 1/2 < \alpha < 1,\\
&\partial w(z),\, \bar{\partial} w(z) = O(|z|^{-1}(\log(1/|z|))^{-2}) && \text{if } \alpha = 1,
\end{aligned}$$

when z approaches 0. In addition, concerning the second partial derivatives,

$$\partial^2 v(z),\, \partial\bar{\partial} v(z) \text{ and } \bar{\partial}^2 v(z) \text{ are continuous at } z = 0 \qquad \text{if } \alpha \le 0;$$

and

$$\begin{aligned}
&\partial^2 v(z),\, \partial\bar{\partial} v(z),\, \bar{\partial}^2 v(z) = O(|z|^{-2\alpha}) && \text{if } 0 < \alpha < 1,\\
&\partial^2 w(z),\, \partial\bar{\partial} w(z),\, \bar{\partial}^2 w(z) = O(|z|^{-2}(\log(1/|z|))^{-2}) && \text{if } \alpha = 1,
\end{aligned} \tag{3.1.1}$$

when z tends to 0.

Our result as follows gives estimates for higher order derivatives of $v(z)$, $w(z)$ near the singularity, and then improve the estimate of the mixed derivatives in Theorem C when the order $\alpha = 1$.

Theorem 3.1.1 ([36])**.** *Let $\kappa(z)$, $u(z)$, $v(z)$, $w(z)$ and α be the same as in Theorem C. If, in addition, $\kappa \in C^{n-2,\nu}(\mathbb{D}^*)$ for an integer $n \ge 2$, and a real number $0 < \nu \le 1$, then for n_1, $n_2 \ge 1$, $n_1 + n_2 = n$, near the origin, $v(z)$ and $w(z)$ satisfy*

$$\begin{aligned}
&\partial^n v(z),\, \bar{\partial}^n v(z),\, \bar{\partial}^{n_1}\partial^{n_2} v(z) = O(|z|^{2-2\alpha-n}) && \text{if } 0 < \alpha < 1;\\
&\partial^n w(z),\, \bar{\partial}^n w(z) = O\left(|z|^{-n}(\log(1/|z|))^{-2}\right) && \text{if } \alpha = 1,\\
&\bar{\partial}^{n_1}\partial^{n_2} w(z) = O\left(|z|^{-n}(\log(1/|z|))^{-3}\right) && \text{if } \alpha = 1.
\end{aligned} \tag{3.1.2}$$

We will show in Chapter 4 that our result is sharp by examining the generalized hyperbolic metric $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ on the thrice-punctured sphere $\widehat{\mathbb{C}}\backslash\{0,1,\infty\}$ with singularities of order α, β, γ at $0, 1, \infty$ when $0 < \alpha, \beta < 1$, $0 < \gamma \le 1$ and $\alpha + \beta + \gamma > 2$ (see [21] for the formula of $\lambda_{\alpha,\beta,\gamma}(z)|dz|$). Theorem 3.1.1 can be divided into two part, case $0 < \alpha < 1$ and case $\alpha = 1$, which will be proved in Section 2.2 and 2.3 respectively.

3.2 Case $0 < \alpha < 1$

Proof of Theorem 3.1.1 when $0 < \alpha < 1$. Theorem 2.3.7 shows that $u \in C^{n,\nu}(\mathbb{D}^*)$, $0 < \nu \le 1$. By Lemma B, for $z \in \mathbb{D}_r^*$, $0 < r < 1$, we have

$$v(z) = h(z) + \frac{1}{2\pi}\int_{\mathbb{D}_r} L(z-\zeta) f(\zeta)\, d\sigma_\zeta, \tag{3.2.3}$$

where

$$f(z) = q(z)|z|^{-2\alpha}, \quad q(z) = -\kappa(z)e^{2v(z)} \tag{3.2.4}$$

and h is a harmonic function on $\mathbb{D}_r$. Now fix $R \in (0,1)$, choose $z \in \mathbb{D}_{R/2}^*$ and let $r = |z|/2$.

Then for a multi-index $\boldsymbol{j}$, $|\boldsymbol{j}| = n \geq 3$, write $z = x_1 + ix_2$, with the same symbols $\boldsymbol{\theta}_\tau$, $\boldsymbol{\phi}_\tau$ as in (2.3.34), (3.2.3) and (2.3.34) lead to

$$\begin{aligned}
& \partial^{\boldsymbol{j}} v(z) \\
= \ & \partial^{\boldsymbol{j}} h(z) + \frac{1}{2\pi}\int_{\mathbb{D}_R\setminus\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta) d\sigma_\zeta + \frac{1}{2\pi}\int_{\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta)\left(f(\zeta) - f(z)\right) d\sigma_\zeta \\
& + \frac{1}{2\pi}\int_{\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta) \sum_{1\leq|\boldsymbol{a}|\leq n} \frac{(\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}} f(z)}{\boldsymbol{a}!} d\sigma_\zeta \\
& - \frac{1}{2\pi}\sum_{\tau=1}^{n-1}\int_{\partial\mathbb{D}_R} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta) \cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](z,\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_{\tau+1}\rangle |d\zeta| \qquad (3.2.5)
\end{aligned}$$

where h is a harmonic function on $\mathbb{D}_R$.

It is known that [10, p. 17],

$$\left|\partial^{\boldsymbol{j}} L(z-\zeta)\right| \leq \frac{n!}{|z-\zeta|^n}. \qquad (3.2.6)$$

Denote $M = \sup_{\zeta\in\mathbb{D}_R} |q(\zeta)|$ and let $C_m > 0$, $m \in \mathbb{N}$, be a constant. Then by (3.2.4),

$$\left|\int_{\mathbb{D}_R\setminus\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta) d\sigma_\zeta\right| \leq M \int_{\mathbb{D}_R\setminus\mathbb{D}_r} \frac{n!}{|z-\zeta|^n}\frac{1}{|\zeta|^{2\alpha}} d\sigma_\zeta \leq \frac{C_1}{|z|^{2\alpha+n-2}},$$

and

$$\begin{aligned}
& \left|\int_{\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta)\left(f(\zeta) - f(z)\right) d\sigma_\zeta\right| \\
\leq \ & \int_{\mathbb{D}_r} \frac{n!}{|z-\zeta|^n}\frac{|q(\zeta) - q(z)|}{|\zeta|^{2\alpha}} d\sigma_\zeta + M\int_{\mathbb{D}_r} \frac{n!}{|z-\zeta|^n}\frac{(|\zeta|^\alpha + |z|^\alpha)||\zeta|^\alpha - |z|^\alpha|}{|z|^{2\alpha}|\zeta|^{2\alpha}} d\sigma_\zeta \\
\leq \ & \frac{C_2}{|z|^{2\alpha+n-2}}.
\end{aligned}$$

That means (3.2.5) is true if $j_1 = 0$ or $j_2 = 0$. If neither j_1 nor j_2 is zero, the first three integrals in (3.2.5) are canceled, and we have to consider the last integral in (3.2.5). In the last sum in (3.2.5) letting $\tau = 1$, we get the following term

$$\partial^{\boldsymbol{\phi}_1} f(z) \cdot \int_{\partial\mathbb{D}_R} \partial^{\boldsymbol{e}_1} L(z-\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_2\rangle |d\zeta|.$$

Writing $\zeta = Re^{i\theta}$ and taking $\boldsymbol{e}_1 = (1,0)$, $\boldsymbol{e}_2 = (0,1)$, without loss of generality, we have

$$\begin{aligned}
& \left|\int_{\partial\mathbb{D}_R} \partial_1 L(z-\zeta) N_2(\zeta) |d\zeta|\right| = \left|\int_{\partial\mathbb{D}_R} \frac{x_1 - R\cos\theta}{|z-\zeta|^2}\sin\theta |d\zeta|\right| \\
= \ & \left|\int_0^{2\pi} \frac{x_1 - R\cos\theta}{|z-\zeta|^2} R\sin\theta d\theta\right| \leq 2\pi\frac{|x_1 - R\cos\theta|}{|z-\zeta|^2} R|\sin\theta| \\
\leq \ & \frac{2\pi R(R+1)}{(R-1)^2} =: C(R). \qquad (3.2.7)
\end{aligned}$$

Note that R is fixed, thus,

$$\left|\int_{\partial\mathbb{D}_R} \partial^{\boldsymbol{e}_1} L(z-\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_2\rangle |d\zeta|\right| \leq C(R)$$

holds for any choice of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Now we consider $\partial^{\phi_1} f(z)$. Since $\partial^{\phi_1} f(z)$ contains the term $q(z) \cdot \partial^{\phi_1}(|z|^{-2\alpha})$ with some coefficient, and

$$\left|\partial^{\phi_1}\left(\frac{1}{|z|^{2\alpha}}\right)\right| \leq \frac{C_3}{|z|^{2\alpha+n-2}},$$

we have

$$\left|q(z)\partial^{\phi_1}\left(\frac{1}{|z|^{2\alpha}}\right) \cdot \int_{\partial\mathbb{D}_r} \partial^{\boldsymbol{e}_1} L(z-\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_2\rangle |d\zeta|\right| \leq \frac{C(R)MC_3}{|z|^{2\alpha+n-2}}.$$

Therefore $\bar{\partial}^{n_1}\partial^{n_2} v = O(|z|^{2-2\alpha-n})$. □

3.3 Case $\alpha = 1$

The proof for $w(z)$ in Theorem 3.1.1 is based on the following lemma.

Lemma D ([19])**.** *Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a continuous function with $\kappa(0) < 0$ and*

$$\kappa(z) = \kappa(0) + O\left(\frac{1}{(\log(1/|z|))^2}\right)$$

as $z \to 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a solution to $\Delta u = -\kappa(z)e^{2u}$ with $u(z) = -\log|z| - \log\log(1/|z|) + w(z)$ where $w(z) = O(1)$ for $z \to 0$, then there exists $0 < \rho < 1$ such that

$$\left|-\kappa(z)e^{2w(z)} - 1\right| \leq \frac{C}{\log(1/|z|)}, \quad z \in \mathbb{D}_\rho^*, \tag{3.3.8}$$

holds for a constant $C > 0$.

Proof of Theorem 3.1.1 when $\alpha = 1$. Theorem 2.3.7 shows that $u(z) \in C^{n,\nu}(\mathbb{D}^*)$. For $w(z)$ defined as in Theorem C, we first show that

$$w(z) = h(z) + \frac{1}{2\pi}\int_{\mathbb{D}_r} L(z-\zeta)\frac{-\kappa(\zeta)e^{2w(\zeta)} - 1}{|\zeta|^2(\log(1/|\zeta|))^2} d\sigma_\zeta \tag{3.3.9}$$

for $z \in \mathbb{D}_r^*$, $0 < r < 1$, where h is harmonic on $\mathbb{D}_r$. Let

$$t(z) := -\log\log(1/|z|), \quad p(z) := w(z) + t(z) = u(z) + \log|z|, \quad z \in \mathbb{D}_r^*.$$

Since

$$\Delta p(z) = -\kappa(z)e^{2u} = \frac{-\kappa(z)e^{2w(z)}}{|z|^2(\log(1/|z|))^2} > 0,$$

$p(z)$ is subharmonic on $\mathbb{D}_r^*$, and $\lim_{z\to 0} p(z) = +\infty$, then $p(z)$ is subharmonic on $\mathbb{D}_r$. As $\Delta p(z)$ is integrable over $\mathbb{D}_r$, by Lemma B,

$$p(z) = h_p(z) + \frac{1}{2\pi}\int_{\mathbb{D}_r} L(z-\zeta)\frac{-\kappa(\zeta)e^{2w(\zeta)}}{|\zeta|^2(\log(1/|\zeta|))^2} d\sigma_\zeta, \quad z \in \mathbb{D}_r,$$

where $h_p(z)$ is harmonic on $\mathbb{D}_r$. For $t(z)$, we have

$$t(z) = h_t(z) + \frac{1}{2\pi}\int_{\mathbb{D}_r} L(z-\zeta)\frac{1}{|\zeta|^2(\log(1/|\zeta|))^2} d\sigma_\zeta, \quad z \in \mathbb{D}_r,$$

where $h_t(z)$ is harmonic on $\mathbb{D}_r$. Setting $w(z) = p(z) - t(z)$ gives (3.3.9) with $h(z) = h_p(z) - h_t(z)$.

Now set $R < 1/e^2$. By Lemma D there exists a number $\rho > 0$ such that the inequality (3.3.8) holds in the disk $\mathbb{D}_\rho$. Let $\widetilde{\rho} = \min\{R/2, \rho\}$. We choose $z \in \mathbb{D}_{\widetilde{\rho}}$ and set $r = |z|/2$. Let

$$q(z) = -\kappa(z)e^{2w(z)} - 1, \quad f(z) = q(z)|z|^{-2}(\log(1/|z|))^{-2}. \tag{3.3.10}$$

Then from (2.3.34), we have

$$\begin{aligned}
& \partial^{\boldsymbol{j}} w(z) \\
= \; & \partial^{\boldsymbol{j}} h(z) + \frac{1}{2\pi}\int_{\mathbb{D}_{\widetilde{\rho}}\setminus\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta) d\sigma_\zeta + \frac{1}{2\pi}\int_{\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta)\left(f(\zeta) - f(z)\right) d\sigma_\zeta \\
& + \frac{1}{2\pi}\int_{\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta) \sum_{1\le|\boldsymbol{a}|\le n} \frac{(\zeta - z)^{\boldsymbol{a}} \partial^{\boldsymbol{a}} f(z)}{\boldsymbol{a}!}\, d\sigma_\zeta \\
& - \frac{1}{2\pi}\sum_{\tau=1}^{n-1} \int_{\partial\mathbb{D}_{\widetilde{\rho}}} \partial^{\boldsymbol{\theta}_\tau} L(z-\zeta) \cdot P_{\tau-1}[\partial^{\boldsymbol{\phi}_\tau} f](z,\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_{\tau+1}\rangle |d\zeta|
\end{aligned} \tag{3.3.11}$$

for a harmonic function h on $\mathbb{D}_{\widetilde{\rho}}$. We obtain

$$\left| \int_{\mathbb{D}_{\widetilde{\rho}}\setminus\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta) f(\zeta) d\sigma_\zeta \right| \le \frac{C_4}{|z|^n(\log(1/|z|))^2},$$

$$\left| \int_{\mathbb{D}_r} \partial^{\boldsymbol{j}} L(z-\zeta)\left(f(\zeta) - f(z)\right) d\sigma_\zeta \right| \le \frac{C_5}{|z|^n(\log(1/|z|))^2},$$

by (3.2.6) and Theorem 3.2.6 in [19]. Thus $\bar{\partial}^n w(z)$, $\partial^n w(z) = O(|z|^{-n}(\log(1/|z|))^{-2})$. For the mixed partial derivatives, the first three integrals in (3.3.11) are canceled, it is sufficient to estimate the last integral in (3.3.11). Letting $\tau = 1$ in the last sum of (3.3.11), the term

$$\partial^{\boldsymbol{\phi}_1} f(z) \int_{\partial\mathbb{D}_{\widetilde{\rho}}} \partial^{\boldsymbol{e}_1} L(z-\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_2\rangle |d\zeta|$$

appears. From (3.3.10) we know that $\partial^{\boldsymbol{\phi}_1} f(z)$ contains $q(z)\cdot\partial^{\boldsymbol{\phi}_1}\left(|z|^{-2}(\log(1/|z|))^{-2}\right)$ with some coefficient. Then it is easy to estimate

$$\left| \partial^{\boldsymbol{\phi}_1} \frac{1}{|z|^2(\log(1/|z|))^2} \right| \le \frac{C_6}{|z|^n(\log(1/|z|))^2}.$$

Thus

$$\begin{aligned}
& \left| q(z)\partial^{\boldsymbol{\phi}_1} \frac{1}{|z|^2(\log(1/|z|))^2} \cdot \partial^{\boldsymbol{e}_1} L(z-\zeta) \cdot \langle N(\zeta), \boldsymbol{e}_2\rangle |d\zeta| \right| \\
\le \; & \frac{2\pi C_7 \cdot C_6}{|z|^n(\log(1/|z|))^3} \frac{\widetilde{\rho}(\widetilde{\rho}+1)}{(\widetilde{\rho}-1)^2},
\end{aligned}$$

provided (3.2.6) and (3.2.7). Therefore $\bar{\partial}^{n_1}\partial^{n_2} w(z) = O(|z|^{-n}(\log(1/|z|))^{-3})$ as desired. □

The second order derivative of $w(z)$ in Theorem C is contained in Theorem 3.1.1. However, for the mixed partial derivative, the estimate (3.1.2) is more accurate than (3.1.1). We take it to be the corollary as following.

Corollary 3.3.1. *Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a locally Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a C^2-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$ with the order $\alpha = 1$ at the point $z = 0$, then for the remainder function $w(z)$ as in Theorem C, the second partial derivatives satisfy*

$$\partial\bar{\partial}w(z) = O(|z|^{-2}(\log(1/|z|))^{-3}).$$

Chapter 4

Generalized hyperbolic metrics

The explicit formula for the generalized hyperbolic metric $\lambda_{\alpha,\,\beta,\,\gamma}(z)|dz|$ on the thrice-punctured sphere $\widehat{\mathbb{C}}\backslash\{0,1,\infty\}$ with the singularities of order α,β,γ when $0<\alpha,\beta<1$, $0<\gamma\leq 1$ at $0,1,\infty$ was given by Kraus, Roth and Sugawa in [21]. In this chapter we investigate the asymptotic properties of the higher order derivatives of $\lambda_{\alpha,\beta,\gamma}(z)$ near the origin and give more precise description for the asymptotic behavior of the remainder function $u(z)$, $w(z)$ of $\log\lambda_{\alpha,\,\beta,\,\gamma}(z)$, and use it to verify the sharpness of Theorem 3.1.1.

4.1 Introduction

From the conformal invariancy of the Gaussian curvature, we know that, under a Möbius transformation which takes $0,1,\infty$ to z_1,z_2,z_3, respectively, the pullback of the hyperbolic metric on $\widehat{\mathbb{C}}\backslash\{z_1,z_2,z_3\}$ is the hyperbolic metric $\lambda_{\alpha,\,\beta,\,\gamma}(z)|dz|$ on $\widehat{\mathbb{C}}\backslash\{0,1,\infty\}$. The following result gives the explicit formula of the hyperbolic metric on twice-punctured plane $\mathbb{C}\backslash\{0,\,1\}$, by using the hypergeometric functions.

Theorem E ([21]). *Let $0<\alpha,\ \beta<1$ and $0<\gamma\leq 1$ such that $\alpha+\beta+\gamma>2$. Then the generalized hyperbolic density on the thrice-punctured sphere $\widehat{\mathbb{C}}\backslash\{0,\,1,\,\infty\}$ of orders $\alpha,\ \beta,\ \gamma$ at $0,\ 1,\ \infty$, respectively, can be expressed by*

$$\begin{aligned}
&\lambda_{\alpha,\,\beta,\,\gamma}(z)\\
=\;& \frac{1}{|z|^{\alpha}|1-z|^{\beta}}\cdot\frac{K_3}{K_1|\varphi_1(z)|^2+K_2|\varphi_2(z)|^2+2Re(\varphi_1(z)\varphi_2(\bar z))} && (4.1.1)\\
=\;& \frac{1}{|z|^{\alpha}|1-z|^{\beta}}\cdot\frac{\delta(1-\alpha)}{|\varphi_1(z)|^2-\delta^2|1-z|^{2-2\alpha}|\varphi_3(z)|^2} && (4.1.2)
\end{aligned}$$

in the twice-punctured plane $\mathbb{C}\backslash\{0,\,1\}$, where

$$K_1:=-\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)},\quad K_2:=-\frac{\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(1-c)\Gamma(a+b+1-c)},\tag{4.1.3}$$

$$K_3 := \sqrt{\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi(c-a))\sin(\pi(c-b))}} \cdot \frac{\Gamma(a+b+1-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}$$

and

$$\varphi_1(z) = F(a,b,c;z), \quad \varphi_2(z) = F(a,b,a+b-c+1;1-z),$$

$$\varphi_3(z) = F(a-c+1, b-c+1, 2-c; z),$$

with

$$a = \frac{\alpha+\beta-\gamma}{2}, \; b = \frac{\alpha+\beta+\gamma-2}{2}, \; c = \alpha; \tag{4.1.4}$$

$$\delta = \frac{\Gamma(c)}{\Gamma(2-c)} \left(\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \right)^{1/2}. \tag{4.1.5}$$

The Gaussian curvature of $\lambda(z)$ defined by (4.1.1) and (4.1.2) is -4. The necessity of the the condition $\alpha+\beta+\gamma > 2$ comes from the Gauss-Bonnet theorem, and the sufficiency is a special case of the classical Schwarz-Picard problem solved by [27] and Heins [11] (see also McOwen [23], Troyanov [32]). Note that φ_1 and φ_3 are analytic in $\mathbb{C}\backslash[1,+\infty)$, φ_2 is analytic in $\mathbb{C}\backslash(-\infty,0]$. We denote $\lambda(z) := \lambda_{\alpha,\beta,\gamma}(z)$ briefly and let $u(z) := \log\lambda(z)$. We will verify the sharpness of Theorem 3.1.1 for corners in Section 3.2 and for cusps in Section 3.3.

4.2 For corners

In this section we consider $\lambda(z)|dz|$ on $\widehat{\mathbb{C}}\backslash\{0,1,\infty\}$ when the order $0<\alpha<1$. The graph of the density is shown in Figure 4.1 when $(\alpha,\beta,\gamma) = (0.6, 0.6, 0.9)$. For the density function

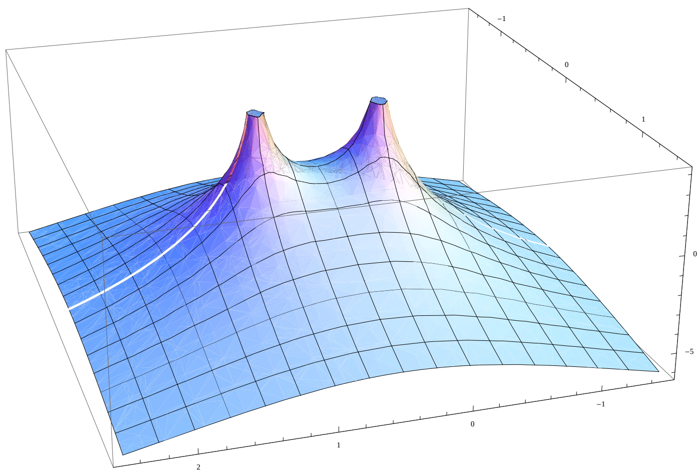

Fig 4.1. Graph of $\lambda_{0.6,0.6,0.9}(z)$

$\lambda(z)$, we can only consider the asymptotic behavior near the origin. By the expression of $\lambda(z)$, we know that the singularity $z=1$ is the same as the origin. As for the infinity, we can change the coordinates by a conformal function, say, $z \mapsto 1/z$, to map ∞ to 0. But some calculation is involved, so it is enough to consider it in a neighborhood of the origin. In the expression (4.1.1), for orders $0 < \alpha,\ \beta < 1$ and $0 < \gamma \leq 1$, the real parameters α, β, γ given by condition (4.1.4) satisfy

$$-\frac{1}{2} < a < 1, \quad -1 < b < /\frac{1}{2}, \quad 0 < c < 1.$$

At first we give a lemma for the future use.

Lemma 4.2.1. *In the expression for $\lambda(z)$ as in (4.1.1) with order $\alpha \in (0,\ 1)$, let*

$$\begin{aligned} M(z) : &= K_1|\varphi_1(z)|^2 + K_2|\varphi_2(z)|^2 + 2Re\,(\varphi_1(z)\varphi_2(\bar{z})) \\ &= (K_1\varphi_1(\bar{z}) + \varphi_2(\bar{z}))\varphi_1(z) + (K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\varphi_2(z). \end{aligned} \tag{4.2.6}$$

Then for a, b and c are defined in (4.1.4), K_1 and K_2 are defined in (4.1.3),

(i) $\lim_{z\to 0} \partial M(z) = \frac{ab}{c}\left(K_1 - \frac{1}{K_2}\right)$ *for* $0 < \alpha < 1/2$,

(ii) $\partial M(z) = 2ab\left(K_1 - \frac{1}{K_2}\right) + 2K_2\,(H(a,b,c))^2\,\frac{\bar{z}}{|z|} + O\left(|z|^{\frac{1}{2}}\right)$ *for* $\alpha = 1/2$,

(iii) $\lim_{z\to 0} z^n|z|^{2\alpha-2}\partial^n M(z) = \frac{(-1)^{n-1}(c)_{n-1}K_2}{1-c}\,(H(a,b,c))^2$ *for* $n \geq 2$ *if* $0 < \alpha \leq 1/2$ *and* $n \geq 1$ *if* $1/2 < \alpha < 1$,

(iv) $\lim_{z\to 0} \bar{z}^m z^n|z|^{2\alpha-2}\bar{\partial}^m\partial^n M(z) = (-1)^{n+m}(c)_{n-1}(c)_{m-1}K_2(H(a,b,c))^2$ *for* $m, n \geq 1$, *where*

$$H(a,b,c) = \frac{\Gamma(c)\Gamma(a+b-c+1)}{\Gamma(a)\Gamma(b)}. \tag{4.2.7}$$

Remark 4.2.2. *The relation in (ii) can be expressed by $\partial M(z) = O(1)$. It is easy to see that there is no non-vanishing limit such as in (iii) holds for $n = 1$ and $\alpha = 1/2$, even if it is multiplied by a power of $z/\bar{z}$.*

Proof of Lemma 4.2.1. Since $\varphi_1(z)$, $\varphi_2(z)$ are analytic in $\mathbb{C}\backslash[1,\ +\infty)$, $\mathbb{C}\backslash(-\infty,\ 0]$ respectively, then we have $\overline{\partial^n\varphi_1(z)} = \bar{\partial}^n\,(\varphi_1(\bar{z}))$ for $z \in \mathbb{C}\backslash[1,\ +\infty)$, $\overline{\partial^n\varphi_2(z)} = \bar{\partial}^n\,(\varphi_2(\bar{z}))$ for $z \in \mathbb{C}\backslash(-\infty,\ 0]$. For the relation in (i), we have

$$\partial M(z) = (K_1\varphi_1(\bar{z}) + \varphi_2(\bar{z}))\partial\varphi_1(z) + (K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial\varphi_2(z).$$

From the property (2.2.13),

$$\partial\varphi_1(0) = \frac{ab}{c}, \tag{4.2.8}$$

From (2.2.16),

$$\varphi_2(0) = F(a,b,a+b-c+1;1) = -\frac{1}{K_2},\ \ \varphi_1(0) = 1, \tag{4.2.9}$$

provided that $a + b < a + b - c + 1$, then

$$M(0) = K_1\varphi_1(0) + \varphi_2(0) = K_1 - {K_2}^{-1}. \tag{4.2.10}$$

Now we consider the term $(K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial\varphi_2(z)$, which satisfies $\lim_{z\to 0}(K_2\varphi_2(z) + \varphi_1(z)) =$

0. From (2.2.14), we obtain

$$
\begin{aligned}
&\varphi_2(z) = F(a,b,a+b-c+1;1-z)\\
= \ &\frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(b-c+1)\Gamma(a-c+1)}\varphi_1(z) + z^{1-c}\frac{\Gamma(c-1)}{\Gamma(c)}H(a,b,c)F(b-c+1,a-c+1,2-c;z)\\
= \ &-\frac{1}{K_2}\varphi_1(z) + z^{1-c}\frac{\Gamma(a+b-c+1)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}F(b-c+1,a-c+1,2-c;z)
\end{aligned}
$$

for $|\arg(z)| < \pi$, which means $\varphi_1(z)$ and $\varphi_2(z)$ are related, thus

$$
K_2\varphi_2(z) + \varphi_1(z) = \frac{-K_2 z^{1-c}}{1-c}H(a,b,c)F(b-c+1,a-c+1,2-c;z) \tag{4.2.11}
$$

and

$$
\lim_{z\to 0}\frac{K_2\varphi_2(z)+\varphi_1(z)}{z^{1-c}} = \frac{-K_2}{1-c}H(a,b,c). \tag{4.2.12}
$$

Near the origin, by (2.2.13), for $n \geq 1$, we have

$$
\begin{aligned}
&\partial^n\varphi_2(z) = \overline{\bar{\partial}^n\varphi_2(\bar{z})}\\
= \ &\frac{(a)_n(b)_n}{(a+b-c+1)_n}(-1)^n F(a+n,b+n,a+b-c+1+n;1-z).
\end{aligned} \tag{4.2.13}
$$

From the property (2.2.14),

$$
\begin{aligned}
&F(a+n,b+n,a+b-c+1+n;1-z)\\
= \ &\frac{\Gamma(a+b-c+1+n)\Gamma(1-c-n)}{\Gamma(b-c+1)\Gamma(a-c+1)}F(a+n,b+n,c+n;z)\\
&+z^{1-c-n}\frac{\Gamma(a+b-c+1+n)\Gamma(c+n-1)}{\Gamma(a+n)\Gamma(b+n)}F(b-c+1,a-c+1,2-c-n;z)
\end{aligned}
$$

for $|\arg(z)| < \pi$. Then near the origin, substituting the equation above into (4.2.13), we have

$$
\begin{aligned}
&\partial^n\varphi_2(z)\\
= \ &\frac{(a)_n(b)_n}{(a+b-c+1)_n}(-1)^n\frac{\Gamma(a+b-c+1+n)\Gamma(1-c-n)}{\Gamma(b-c+1)\Gamma(a-c+1)}F(a+n,b+n,c+n;z)\\
&+\frac{(-1)^n}{z^{n+c-1}}\frac{\Gamma(a+b-c+1)\Gamma(c+n-1)}{\Gamma(a)\Gamma(b)}F(b-c+1,a-c+1,2-c-n;z),
\end{aligned} \tag{4.2.14}
$$

which leads to the limit

$$
\lim_{z\to 0} z^{n+c-1}\partial^n\varphi_2(z) = (-1)^n(c)_{n-1}H(a,b,c). \tag{4.2.15}
$$

Letting $n = 1$ in (4.2.14) and combining with (4.2.11), when $0 < c = \alpha < \frac{1}{2}$, we have $\lim_{z\to 0}(K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial\varphi_2(z) = 0$. Thus

$$
\lim_{z\to 0}\partial M(z) = \lim_{z\to 0}(K_1\varphi_1(\bar{z}) + \varphi_2(\bar{z}))\partial\varphi_1(z) = \frac{ab}{c}\left(K_1 - \frac{1}{K_2}\right)
$$

provided (4.2.8) and (4.2.10).

We note that (4.2.11) and (4.2.14) are valid for $n = 1$, $\alpha = 1/2$. Then combining with (4.2.8) and (4.2.9) we obtain the relation in (ii).

For the limit in (iii),

$$\partial^n M(z) = (K_1\varphi_1(\bar{z}) + \varphi_2(\bar{z}))\partial^n\varphi_1(z) + (K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial^n\varphi_2(z). \tag{4.2.16}$$

From the property (2.2.13), we have

$$\partial^n\varphi_1(0) = \frac{(a)_n(b)_n}{(c)_n}, \qquad n \geq 1. \tag{4.2.17}$$

Since $n > 2\alpha - 2$ for all $n \geq 2$ and $0 < \alpha < 1$, from (4.2.17) and (4.2.10), we know that the limit in (iii) is only determined by the term $(K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial^n\varphi_2(z)$. From (4.2.16), (4.2.15) and (4.2.12), we have

$$\begin{aligned}
&\lim_{z\to 0} z^n|z|^{2\alpha-2}\partial^n M(z) = \lim_{z\to 0} z^n|z|^{2\alpha-2}(K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial^n\varphi_2(z) \\
= &\lim_{z\to 0} \frac{K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z})}{\bar{z}^{1-\alpha}} \frac{z^n}{z^{1-\alpha}}\partial^n\varphi_2(z) = \lim_{z\to 0} \frac{K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z})}{\bar{z}^{1-c}} \cdot \lim_{z\to 0} z^{n+c-1}\partial^n\varphi_2(z) \\
= &\frac{(-1)^{n-1}(c)_{n-1}K_2}{1-c}\left(H(a,b,c)\right)^2.
\end{aligned}$$

For the limit in (iv), if $m \geq 1,\ n \geq 1$, we have

$$\begin{aligned}
& \bar{\partial}^m\partial^n M(z) \\
= \ & (K_1\bar{\partial}^m\varphi_1(\bar{z}) + \bar{\partial}^m\varphi_2(\bar{z}))\partial^n\varphi_1(z) + (K_2\bar{\partial}^m\varphi_2(\bar{z}) + \bar{\partial}^m\varphi_1(\bar{z}))\partial^n\varphi_2(z).
\end{aligned}$$

Since $\lim_{z\to 0} z^{n+c-1}\partial^n\varphi_1(z) = 0$, we obtain

$$\begin{aligned}
& \lim_{z\to 0} \bar{z}^m z^n|z|^{2\alpha-2}\bar{\partial}^m\partial^n M(z) \\
= \ & \lim_{z\to 0} \frac{\bar{z}^m z^n}{|z|^{2-2c}} K_2\bar{\partial}^m\varphi_2(\bar{z})\partial^n\varphi_2(z) \\
= \ & \lim_{z\to 0} K_2 \frac{\bar{z}^m}{\bar{z}^{1-c}}\bar{\partial}^m\varphi_2(\bar{z}) \cdot \frac{z^n}{z^{1-c}}\partial^n\varphi_2(z) \\
= \ & (-1)^{m+n}(c)_{m-1}(c)_{n-1}K_2\left(H(a,b,c)\right)^2.
\end{aligned}$$

□

Now we verify the sharpness of Theorem 3.1.1, and also Theorem 1.1 in [19], when $0 < \alpha < 1$.

Theorem 4.2.3. *For m, $n \geq 1$ and λ defined by (4.1.1) with the order $0 < \alpha < 1$, $H(a,b,c)$ given by (4.2.7), near the origin, the remainder function $v(z)$ satisfies*

(i) $\lim_{z\to 0} \partial v(z) = \frac{ab}{c}$ *for* $0 < \alpha < 1/2$,

(ii) $\partial v(z) = 2ab + \frac{2K_2^2}{K_1K_2 - 1}\left(H(a,b,c)\right)^2 \frac{\bar{z}}{|z|} + O\left(|z|^{\frac{1}{2}}\right)$ *near the origin for* $\alpha = 1/2$,

(iii) $\lim_{z\to 0} z^n|z|^{2\alpha-2}\partial^n v(z) = \frac{(-1)^{n-1}(c)_{n-1}K_2^2}{(1-c)(K_1K_2-1)}\left(H(a,b,c)\right)^2$ *for* $n \geq 2$ *if* $0 < \alpha \leq 1/2$ *and*

$n \geq 1$ *if* $1/2 < \alpha < 1$,

(iv) $\lim_{z\to 0} \bar{z}^m z^n |z|^{2\alpha-2} \bar{\partial}^m \partial^n v(z) = \frac{(-1)^{n+m}(c)_{n-1}(c)_{m-1}K_2^2}{K_1K_2-1}(H(a,b,c))^2$ *for* $m,\ n \geq 1$ *and* $0 < \alpha < 1$.

Proof. For $\lambda(z)$ in (4.1.1), $v(z) = -\beta \log|1-z| + \log K_3 - \log M(z)$ and

$$\partial^n \log|1-z| = \frac{-(n-1)!}{2(1-z)^n}, \tag{4.2.18}$$

we may consider $\partial^n \log M(z)$ only. From the expression (4.2.18), the limits $\lim_{z\to 0} \partial v(z)$ and $\lim_{z\to 0} z^n|z|^{2\alpha-2}\partial^n v(z)$ both depend on the term $\partial^n M(z)$. From (4.2.9),

$$M(0) = K_1\varphi_1(0) + \varphi_2(0) = K_1 - \frac{1}{K_2}. \tag{4.2.19}$$

Thus by Lemma 4.2.1 and (4.2.19),

$$\lim_{z\to 0} \partial v(z) = \lim_{z\to 0} \frac{\partial M(z)}{M(z)},$$

$$\lim_{z\to 0} z^n|z|^{2\alpha-2}\partial^n v(z) = \lim_{z\to 0} z^n|z|^{2\alpha-2}\frac{\partial^n M(z)}{M(z)}.$$

Hence we obtain the four cases in Theorem 4.2.3 corresponding to ones in Lemma 4.2.6. □

4.3 For cusps

When $\alpha = 1$, the formula for $\lambda_{1,\beta,\gamma}(z)$ is to be understood in the limit sense $\lim_{\alpha\to 1}$. Thus we have

$$K_3 = \frac{1}{B(a,b)} := \frac{1}{B}, \quad K_2 = 0, \quad S := \frac{\pi \sin(\pi(a+b))}{\sin \pi a \sin \pi b}, \quad K_1 = -\frac{S}{B}, \tag{4.3.20}$$

$$\varphi_1(z) = F(a,b,1;z), \quad \varphi_2(z) = F(a,b,a+b;1-z).$$

And

$$\begin{aligned} \lambda_{1,\beta,\gamma}(z) &= \frac{1}{|z|}\frac{1}{|1-z|^\beta}\frac{K_3}{K_1|\varphi_1(z)|^2 + \varphi_1(z)\varphi_2(\bar{z}) + \varphi_1(\bar{z})\varphi_2(z)} \\ &:= \frac{1}{|z|}\frac{1}{|1-z|^\beta}\frac{K_3}{M(z)}. \end{aligned} \tag{4.3.21}$$

The remainder function of $u(z)$ near the origin is

$$w(z) = -\beta \log|1-z| + \log K_3 - \log M(z) + \log\log(1/|z|). \tag{4.3.22}$$

The assumption of Theorem E and (4.1.4) show that a and b satisfy $0 < a < 1$, $0 < b < 1/2$, $0 < a + b < 1$. The function

$$2R - S = 4\Psi(1) - 2\Psi(a) - 2\Psi(b) - \pi \cot \pi a - \pi \cot \pi b = G(a) + G(b) \tag{4.3.23}$$

will be used frequently. For all $a,\ b$ given by (4.1.4), $2R - S > 0$.

We can verify the sharpness of Theorem 3.1.1 by use of $\lambda_{1,\beta,\gamma}(z)$ given by (4.3.21). Furthermore, for $\lambda(z)$, we can obtain the precise estimate for higher order derivatives of $w(z)$ near the origin. The following result is stronger than Theorem 3.1.1.

Theorem 4.3.1. *Let $\lambda(z) := \lambda_{1,\beta,\gamma}(z)$ given by (4.3.21) with β and γ satisfying the condition in Theorem E, and $w(z)$ be the remainder function of $u(z)$. Then for $m,\ n \geq 1$, we have*

(i) $\lim_{z\to 0} z^n(\log(1/|z|))^2\partial^n w(z) = \dfrac{(-1)^n(n-1)!}{4}(G(a)+G(b))$,

(ii) $\lim_{z\to 0} z^n\bar{z}^m(\log(1/|z|))^3\bar{\partial}^m\partial^n w(z) = \dfrac{(-1)^{m+n-1}(n-1)!(m-1)!}{4}(G(a)+G(b))$,

where the function G is defined by (2.2.19) *and a, b are given by* (4.1.4).

Proof. We first estimate the derivatives of $\log M(z)$. Since

$$\begin{aligned}
\partial^n\varphi_1(z) &= \frac{(a)_n(b)_n}{n!}F(a+n,b+n,n+1;z),\\
\partial^n\varphi_2(z) &= (-1)^n\frac{(a)_n(b)_n}{(a+b)_n}F(a+n,b+n,a+b+n;1-z),
\end{aligned}$$

we have

$$\partial^n\varphi_1(0) = \frac{(a)_n(b)_n}{n!}. \tag{4.3.24}$$

Near the origin, by (2.2.16) we have

$$\partial^n\varphi_2(z) = \frac{(a)_n(b)_n}{(a+b)_n}\frac{(-1)^n}{z^n}F(b,a,a+b+n;1-z). \tag{4.3.25}$$

The expression (2.2.15) shows that

$$F(b,a,a+b+n;1-z) = \frac{\Gamma(a+b+n)\Gamma(n)}{\Gamma(a+n)\Gamma(b+n)} + O(|z|\log|z|)$$

near the origin. Thus (4.3.25) becomes

$$\begin{aligned}
\partial^n\varphi_2(z) &= \frac{(-1)^n}{z^n}\frac{(a)_n(b)_n}{(a+b)_n}\left(\frac{\Gamma(a+b+n)\Gamma(n)}{\Gamma(a+n)\Gamma(b+n)} + O(|z|\log|z|)\right)\\
&= \frac{(-1)^n(n-1)!}{Bz^n} + O\left(\frac{\log|z|}{|z|^{n-1}}\right).
\end{aligned} \tag{4.3.26}$$

The property (2.2.16) gives

$$\varphi_2(z) = \frac{1}{B}\left(\log\frac{1}{z} + R\right)(1+O(z)). \tag{4.3.27}$$

Combining estimates (4.3.24), (4.3.26), (4.3.27), we obtain

$$\begin{aligned}
\partial^n M(z) &= (K_1\varphi_1(\bar{z})+\varphi_2(\bar{z}))\partial^n\varphi_1(z) + \varphi_1(\bar{z})\partial^n\varphi_2(z)\\
&= \frac{(a)_n(b)_n}{B\ n!}\log\frac{1}{\bar{z}} + \frac{(-1)^n(n-1)!}{Bz^n} + O\left(\frac{\log|z|}{|z|^{n-1}}\right)
\end{aligned}$$

and

$$\begin{aligned} M(z) &= K_1\varphi_1(\bar z)\varphi_1(z)+\varphi_2(\bar z)\varphi_1(z)+\varphi_1(\bar z)\varphi_2(z) \\ &= \frac{2\log(1/|z|)}{B}\left(1+\frac{2R-S}{2\log(1/|z|)}+O(|z|)\right) \end{aligned}$$

near the origin. Then

$$\frac{\partial^n M(z)}{M(z)} = \partial^n M(z)\,\frac{B}{2\log(1/|z|)}\left(1-\frac{2R-S}{2\log(1/|z|)}+O(|z|)\right) \tag{4.3.28}$$

$$= \frac{(-1)^n(n-1)!}{2z^n\log(1/|z|)}-\frac{(-1)^n(n-1)!(G(a)+G(b))}{4z^n\log^2(1/|z|)}+O\left(\frac{1}{|z|^{n-1}}\right). \tag{4.3.29}$$

We note that

$$\begin{aligned} &\bar\partial^m\partial^n M(z) = (K_1\bar\partial^m\varphi_1(\bar z)+\bar\partial^m\varphi_2(\bar z))\partial^n\varphi_1(z)+\bar\partial^m\varphi_1(\bar z)\partial^n\varphi_2(z) \\ &= \frac{(-1)^m(m-1)!}{B\bar z^m}\frac{(a)_n(b)_n}{n!}+\frac{(-1)^n(n-1)!}{Bz^n}\frac{(a)_m(b)_m}{m!}+O\left(\frac{\log|z|}{|z|^{m-1}}\right)+O\left(\frac{\log|z|}{|z|^{n-1}}\right), \end{aligned}$$

the same technique as used in (4.3.28) leads to

$$\frac{\bar\partial^m\partial^n M(z)}{M(z)} = O\left(\frac{1}{|z|^\tau\log(1/|z|)}\right), \tag{4.3.30}$$

where $\tau=\max\{m,\, n\}<m+n$.

Now we consider derivatives of $w(z)$. In the pure derivative case,

$$\partial^n w(z) = \frac{\beta(n-1)!}{2(1-z)^n}-\partial^n\log M(z)+\partial^n\log\log(1/|z|). \tag{4.3.31}$$

Note that $\partial^n\log M(z)$ is a linear combination of finitely many terms of the form

$$\prod_{j=1}^{N}\frac{\partial^{n_j}M(z)}{M(z)} \tag{4.3.32}$$

for $1\le N\le n$ and $\sum_{j=1}^k n_j=n$. When $N=1$, the term (4.3.32) is equal to $\frac{\partial^n M(z)}{M(z)}$. By induction we know the coefficient of $\frac{\partial^n M(z)}{M(z)}$ in $\partial^n\log M(z)$ is always 1. Note that the term $z^{-n}\log^{-1}(1/|z|)$ only appears in $\partial^n M(z)/M(z)$, and (2.2.21), (4.3.29), (4.3.31) show that $z^{-n}\log^{-1}(1/|z|)$ is canceled in $\partial^n w(z)$. We should look at the term which contains $z^{-n}\log^{-2}(1/|z|)$ for the higher order derivatives, while the higher power terms in (4.3.29) are ignored for a moment. For the term (4.3.32), the estimate (4.3.29) shows that

$$\prod_{j=1}^{N}\frac{\partial^{n_j}M(z)}{M(z)} = O\left(\frac{1}{z^n\log^N(1/|z|)}\right), \qquad n=\sum_{j=1}^N n_j.$$

Therefore, to generate the $z^{-n}\log^{-2}(1/|z|)$ term, N is at most 2, thus the $z^{-n}\log^{-2}|z|$ term of $\partial^n\log M(z)$ only appears in

$$\frac{\partial^n M}{M}-\frac{1}{2}\sum_{j=1}^{n-1}\binom{n}{j}\frac{\partial^j M\partial^{n-j}M}{M^2}.$$

For every j, $1 \le j \le n-1$, we have

$$\frac{\partial^j M \partial^{n-j} M}{M^2} = \frac{(-1)^n (j-1)!(n-j-1)!}{4z^n \log^2 |z|} + O(\frac{1}{|z|^n \log^3(1/|z|)}).$$

Denote the coefficient of $z^{-n}\log^{-2}|z|$ in $\sum_{j=1}^{n-1}\binom{n}{j}(\partial^j M\partial^{n-j}M/M^2)$ by k_n, then (4.3.29) results in

$$\begin{aligned}
k_n &= \sum_{j=1}^{n-1}\binom{n}{j}\frac{(-1)^n(j-1)!(n-j-1)!}{4} = \frac{(-1)^n(n-1)!}{4}\sum_{j=1}^{n-1}\frac{n}{j(n-j)} \\
&= \frac{(-1)^n(n-1)!}{4}\sum_{j=1}^{n-1}\left(\frac{1}{j}+\frac{1}{n-j}\right) \\
&= \frac{(-1)^n(n-1)!}{2}\sum_{j=1}^{n-1}\frac{1}{j} = -2\mathcal{B}_n.
\end{aligned} \tag{4.3.33}$$

In combination with (2.2.22) and (4.3.33), we obtain

$$\begin{aligned}
\lim_{z\to 0} z^n \log^2(1/|z|)\partial^n w(z) &= \frac{(-1)^n(n-1)!(G(a)+G(b))}{4} + \frac{1}{2}k_n + \mathcal{B}_n \\
&= \frac{(-1)^n(n-1)!}{4}(G(a)+G(b)),
\end{aligned}$$

thus (i) holds.

For the mixed derivatives,

$$\bar\partial^m\partial^n w(z) = -\bar\partial^m\partial^n \log M(z) + \bar\partial^m\partial^n \log\log(1/|z|).$$

It is known that $M(z) = M(\bar z)$ and $\bar\partial^m\partial^n \log M(z) = \overline{\bar\partial^n\partial^m \log M(z)}$. Thus without loss of generality we may assume $m \le n$. Note that the term $\bar z^{-m}z^{-n}\log^{-2}(1/|z|)$ only occurs in $\bar\partial^m M\partial^n M/M^2$ with the coefficient $(-1)^{m+n}(n-1)!(m-1)!/4$, comparing with (2.2.25) shows that there is no term of $\bar z^{-m}z^{-n}\log^{-2}(1/|z|)$ left in the expression for $\bar\partial^m\partial^n w(z)$. Hence the coefficient of $z^{-n}\bar z^{-m}\log^{-3}(1/|z|)$ is desired. The term containing $z^{-n}\bar z^{-m}\log^{-3}(1/|z|)$ is the product of at most three terms in the forms of $\partial^{n_j}M/M$ or $\bar\partial^{m_j}M/M$. Estimate (4.3.30) and (4.3.29) imply that the term $z^{-n}\bar z^{-m}\log^{-3}(1/|z|)$ of $\bar\partial^m\partial^n\log M(z)$ only appears in

$$-\frac{\bar\partial^m M\partial^n M}{M^2} + \frac{\bar\partial^m M}{M}\sum_{j=1}^{n-1}\binom{n}{k}\frac{\partial^j M\partial^{n-j}M}{M^2} + \frac{\partial^n M}{M}\sum_{j=1}^{m-1}\binom{m}{k}\frac{\bar\partial^j M\bar\partial^{m-j}M}{M^2}$$

for $m \ge 1$, $n \ge 1$. Thus

$$\bar\partial^m\partial^n \log M(z) = \frac{t_{m,n}}{z^n\bar z^m\log^3(1/|z|)} + O\left(\frac{1}{|z|^{m+n+1}\log^3(1/|z|)}\right)$$

with

$$t_{m,n} = \left(G(a)+G(b)+\sum_{j=1}^{n-1}\frac{1}{j}+\sum_{j=1}^{m-1}\frac{1}{j}\right)\frac{(-1)^{m+n}(m-1)!(n-1)!}{4}, \tag{4.3.34}$$

which can be obtained by the technique similar as applied to obtain k_n in (4.3.33). Thus for

$m \geq 1,\ n \geq 1$, by (2.2.26) and (4.3.34), we obtain

$$
\begin{aligned}
& \lim_{z \to 0} z^n \bar{z}^m \log^3(1/|z|) \bar{\partial}^m \partial^n w(z) \\
= \quad & -t_{m,n} + \mathcal{D}_m = \frac{(-1)^{m+n-1}(n-1)!(m-1)!}{4}(G(a) + G(b))
\end{aligned}
$$

This completes the proof and verifies the sharpness of Theorem 3.1.1. $\square$

Chapter 5

Extension of Minda's work

Minda [24] investigated the precise limiting behavior of the hyperbolic density function in a neighborhood of a puncture on the plane domain using the uniformisation theorem in 1997. He also made a connection between the hyperbolic density and the Schwarzian, pre-Schwarzian derivatives. Kraus and Roth [19] extended Minda's work to a conformal metric with negative curvature in 2008. In this chapter we show the higher order limiting behavior of a conformal metric with negative curvature κ when κ is smooth enough.

5.1 Introduction

The following theorem is due to Minda.

Theorem F ([24]). *Suppose Ω is a hyperbolic region in the complex plane $\mathbb{C}$, $c \in \mathbb{C}$ is an isolated boundary point of Ω, and $\lambda_\Omega(\omega)|d\omega|$ is the hyperbolic metric on Ω. Then*

$$\lim_{\omega \to c} [|\omega - c| \log(1/|\omega - c|)]\lambda_\Omega(\omega) = 1/2. \tag{5.1.1}$$

Then Kraus and Roth gave an extension of Theorem F as follows.

Theorem G ([19]). *Let $\lambda(z)|dz|$ be a regular conformal metric on $\mathbb{D}^*$ with an isolated singularity at $z = 0$. Suppose that the curvature $\kappa : \mathbb{D}^* \to \mathbb{R}$ has a Hölder continuous extension to $\mathbb{D}$ such that $\kappa(0) < 0$. Then $\log \lambda$ has the order $\alpha \leq 1$ at $z = 0$ and*

$$\lim_{z \to 0} |z| \log(1/|z|)\lambda(z) = \begin{cases} 0 & \text{if } \alpha < 1 \\ \dfrac{1}{\sqrt{-\kappa(0)}} & \text{if } \alpha = 1. \end{cases}$$

Our work is to extend Theorems F and G and to give the limits of higher order derivatives for SK-metrics. As a motivation we have the following limits of up to the third order derivatives of a regular conformal metric on $\mathbb{D}^*$.

Theorem 5.1.1. *Let $\lambda(z)|dz|$ be a regular conformal metric on $\mathbb{D}^*$ with a singularity at $z = 0$ of order 1. Suppose that the curvature $\kappa : \mathbb{D}^* \to \mathbb{R}$ has a Hölder continuous extension* *to $\mathbb{D}$ such that $\kappa(0) < 0$. Then*

(i) $\lim_{z\to 0} z|z|\log(1/|z|)\partial\lambda(z) = -\dfrac{1}{2\sqrt{-\kappa(0)}}$,

(ii) $\lim_{z\to 0} z^2|z|\log(1/|z|)\partial^2\lambda(z) = \dfrac{3}{4\sqrt{-\kappa(0)}}$,

(iii) $\lim_{z\to 0} |z|^3\log(1/|z|)\partial\bar\partial\lambda(z) = \dfrac{1}{4\sqrt{-\kappa(0)}}$.

In a domain Ω with the conformal metric $\lambda(\omega)|d\omega|$, if the boundary point $c \in \bar\Omega$ is a corner of order α, the analogous limit of (5.1.1), defined by

$$l' := \lim_{\omega\to c} |\omega - c|^\alpha \lambda(w), \tag{5.1.2}$$

exists near c by Theorem C, but l' cannot necessarily expressed in terms of α only. In Section 4.2 we prove Theorem 5.1.1 and give its extension for SK-metrics when the singularity is a cusp. In Section 4.3 we provide the limits for the generalized hyperbolic metric $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ on $\mathbb{C}\backslash\{0,1\}$ when the order $\alpha \in (0,1)$, as a special case that l' can be given by the orders α, β and γ.

5.2 Limits of higher order derivatives near cusps

Proof of Theorem 5.1.1. Let $u(z) := \log\lambda(z)$, such that $\partial\lambda(z) = \partial u(z)\lambda(z)$. Noting that $u(z) = -\alpha\log|z| + v(z)$ if $\alpha < 1$, $u(z) = -\log|z| - \log\log(1/|z|) + w(z)$ if $\alpha = 1$, by Theorem C, we have

$$\lim_{z\to 0} z\partial u(z) = -\frac{1}{2}, \quad \lim_{z\to 0} z^2\partial^2 u(z) = \frac{1}{2}, \quad \lim_{z\to 0} |z|^2\partial\bar\partial u(z) = 0.$$

In combination with Theorem G, we have

$$\begin{aligned} & \lim_{z\to 0} z|z|\log(1/|z|)\partial\lambda(z) = \lim_{z\to 0} z|z|\log(1/|z|)\partial u(z)\lambda(z) \\ = \quad & \lim_{z\to 0} |z|\log(1/|z|)\lambda(z)\cdot z\partial u(z) = -\frac{1}{2\sqrt{-\kappa(0)}} \end{aligned}$$

for the limit in (i),

$$\begin{aligned} & \lim_{z\to 0} z^2|z|\log(1/|z|)\partial^2\lambda(z) = \lim_{z\to 0} z^2|z|\log(1/|z|)\left(\partial^2 u(z)\lambda(z) + \partial u(z)\partial\lambda(z)\right) \\ = \quad & \lim_{z\to 0} z^2\partial^2 u(z)\cdot|z|\log(1/|z|)\lambda(z) + \lim_{z\to 0} z|z|\log(1/|z|)\partial\lambda(z)\cdot z\partial u(z) \\ = \quad & \frac{1}{2\sqrt{-\kappa(0)}} + (-\frac{1}{2})\cdot(-\frac{1}{2\sqrt{-\kappa(0)}}) = \frac{3}{4\sqrt{-\kappa(0)}} \end{aligned}$$

for the limit in (ii), and

$$\begin{aligned} & \lim_{z\to 0} |z|^3\log(1/|z|)\partial\bar\partial\lambda(z) = \lim_{z\to 0} |z|^3\log(1/|z|)\left(\partial\bar\partial u(z)\lambda(z) + \partial u(z)\bar\partial\lambda(z)\right) \\ = \quad & \lim_{z\to 0} |z|^2\partial\bar\partial u(z)\cdot|z|\log(1/|z|)\lambda(z) + \lim_{z\to 0} \bar z|z|\log(1/|z|)\bar\partial\lambda(z)\cdot z\partial u(z) \\ = \quad & -\frac{1}{2\sqrt{-\kappa(0)}}\cdot(-\frac{1}{2}) = \frac{1}{4\sqrt{-\kappa(0)}} \end{aligned}$$

for the limit in (iii). $\square$

Theorem 5.1.1 is given for a regular conformal metric $\lambda(\omega)|d\omega|$ with a (locally) Hölder continuous Gaussian curvature κ. Considering Theorems C and 3.1.1, if we add the assumption that κ is $(n-2)$-th order (locally) Hölder continuous, we can obtain the following higher order version of Theorems 5.1.1 for $\log \lambda(z)$ in $\mathbb{D}^*$.

Theorem 5.2.1. *Let $\kappa : \mathbb{D} \to \mathbb{R}$ be of class $C^{n-2,\nu}(\mathbb{D})$ for an integer $n \geq 3$, $0 < \nu \leq 1$ and $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a $C^{n,\nu}$-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$, then u has order $\alpha \in (-\infty, 1]$ and for n_1, $n_2 \geq 1$, $n_1 + n_2 \leq n$,*

(i) $\lim_{z\to 0} z^n \partial^n u(z) = \dfrac{\alpha}{2}(-1)^n(n-1)! = \lim_{z\to 0} \bar{z}^n \bar{\partial}^n u(z)$,

(ii) $\lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} u(z) = 0$.

Proof. When $0 < \alpha < 1$, $u(z) = -\alpha \log|z| + v(z)$. Theorems 3.1.1 implies that

$$\lim_{z\to 0} z^n \partial^n v(z) = 0, \quad \lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} v(z) = 0.$$

Since

$$\partial^n \log|z| = \frac{(-1)^{n-1}(n-1)!}{2z^n}, \quad \bar{\partial}^{n_1} \partial^{n_2} \log|z| = 0, \tag{5.2.3}$$

Then we have

$$\lim_{z\to 0} z^n \partial^n u(z) = -\alpha \lim_{z\to 0} z^n \partial^n \log|z| + \lim_{z\to 0} z^n \partial^n v(z) = \frac{\alpha}{2}(-1)^n(n-1)!,$$

$$\lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} u(z) = 0.$$

When $\alpha = 1$, $u(z) = -\log|z| - \log\log(1/|z|) + w(z)$. From Theorems 4.1 and 4.3 we have

$$\lim_{z\to 0} z^n \partial^n w(z) = 0, \quad \lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} w(z) = 0.$$

By (2.2.20) and (2.2.23) we obtain that

$$\lim_{z\to 0} z^n \partial^n \log\log(1/|z|) = 0, \quad \lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} \log\log(1/|z|) = 0. \tag{5.2.4}$$

Combining (5.2.4) with (5.2.3) leads to

$$\lim_{z\to 0} z^n \partial^n u(z) = -\alpha \lim_{z\to 0} z^n \partial^n \log|z| + \lim_{z\to 0} z^n \partial^n v(z) = \frac{(-1)^n(n-1)!}{2},$$

$$\lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} u(z) = 0. \qquad \square$$

Remark 5.2.2. *From* (2.2.24) *and* (2.2.25)*, for $n_1, n_2 \leq 1$,*

$$\lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \left(\log\frac{1}{|z|}\right)^2 \bar{\partial}^{n_1} \partial^{n_2} \log\log\frac{1}{|z|} = \mathcal{C}_1,$$

thus, from (3.1.2) *we can obtain a limit stronger than the second limit in for the mixed* *derivative of $u(z)$ when the order $\alpha = 1$,*

$$\lim_{z\to 0} \bar{z}^{n_1} z^{n_2} \left(\log \frac{1}{|z|}\right)^2 \bar{\partial}^{n_1} \partial^{n_2} u(z) = \mathcal{C}_1 = \frac{(-1)^{n_1+n_2-1}}{4} (n_1-1)!(n_2-1)!$$

(see [35]).

Based on Theorem 5.2.1, we can provide the following result as an estimate for the higher order derivatives of a conformal metric with the negative curvature near the origin when $\alpha = 1$.

Theorem 5.2.3. *Let κ and u be the same as in Theorem 5.2.1. If the order of u is $\alpha = 1$, then for n_1, $n_2 \geq 0$, $n_1 + n_2 \leq n$, the limit*

$$l_{n_1,n_2} := \frac{1}{n_1! n_2!} \lim_{z\to 0} |z| \log(1/|z|) \bar{z}^{n_1} z^{n_2} \bar{\partial}^{n_1} \partial^{n_2} \lambda(z)$$

exists. Moreover, l_{n_1,n_2} satisfies the following
(i) $l_{n_1,n_2} = \binom{-\frac{1}{2}}{n_1}\binom{-\frac{1}{2}}{n_2} \frac{1}{\sqrt{-\kappa(0)}}$,

(ii) $l_{n_1,n_2} = l_{n_2,n_1}$,
where

$$\binom{\tau}{j} = \frac{\tau(\tau-1)\cdots(\tau-j+1)}{j\,!}$$

is the binomial coefficient.

Proof. From Theorem 3.1.1 and (2.2.20), (2.2.23), we can obtain the existence of l_{n_1,n_2}.
We write $\lambda(z) = e^{u(z)}$, then $\partial\lambda(z) = \lambda(z)\,\partial u(z)$, and

$$\partial^n \lambda(z) = \sum_{j=0}^{n-1} \binom{n-1}{j} \partial^{n-j} u(z)\, \partial^j \lambda(z)$$

by induction, where $\partial^0\lambda(z) = \bar{\partial}^0\lambda(z) = \lambda(z)$. Thus

$$l_{0,n_2} = \frac{1}{n_2!} \lim_{z\to 0} \sum_{j=0}^{n_2-1} \binom{n_2-1}{j} z^{n_2-j} \partial^{n_2-j} u(z) \cdot |z| \log(1/|z|) z^j \partial^j \lambda(z).$$

The limit (ii) in Theorem 5.2.1 enables us to write l_{n_1,n_2} as a sum of the terms only containing pure derivatives of $u(z)$,

$$\begin{aligned} & l_{n_1,n_2} \\ = \; & \frac{1}{n_1! n_2!} \lim_{z\to 0} \sum_{j=0}^{n_2-1} \binom{n_2-1}{j} z^{n_2-j} \partial^{n_2-j} u(z)\, |z| \log(1/|z|) \bar{z}^{n_1} z^j \bar{\partial}^{n_1} \partial^j \lambda(z). \end{aligned} \tag{5.2.5}$$

By Theorem 5.1.1, it is known that $l_{0,1}$ is a real number, so $l_{1,0} = \overline{l_{0,1}} = l_{0,1}$. Since

$$\bar{\partial}^{n_2} \lambda(z) = \sum_{j=0}^{n_2-1} \binom{n_2-1}{j} \bar{\partial}^{n_2-j} u(z)\, \bar{\partial}^j \lambda(z), \tag{5.2.6}$$

by induction, $l_{n_2,0} = l_{0,n_2}$. From (5.2.5), (5.2.6), and (i) of Theorem 5.2.1, we have

$$\begin{aligned} & l_{n_1,n_2} \\ = & \sum_{j=0}^{n_2-1} \lim_{z\to 0} \frac{1}{n_1! n_2!} \frac{(n_2-1)!}{j!(n_2-1-j)!} z^{n_2-j}\partial^{n_2-j}u(z)\cdot |z|\log(1/|z|)\bar{z}^{n_1}z^j\bar{\partial}^{n_1}\partial^j\lambda(z) \\ = & \frac{1}{n_2}\sum_{j=0}^{n_2-1} \frac{1}{n_1!}\frac{1}{j!(n_2-1-j)!} \lim_{z\to 0} z^{n_2-j}\partial^{n_2-j}u(z)\cdot \lim_{z\to 0}|z|\log(1/|z|)\bar{z}^{n_1}z^j\bar{\partial}^{n_1}\partial^j\lambda(z) \\ = & \frac{1}{n_2}\sum_{j=0}^{n_2-1} \frac{(-1)^{n_2-j}}{2}\frac{1}{n_1! j!} \lim_{z\to 0}|z|\log(1/|z|)\bar{z}^{n_1}z^j\bar{\partial}^{n_1}\partial^j\lambda(z) \\ = & \frac{1}{2n_2}\sum_{j=1}^{n_2-1}(-1)^{n_2-j} l_{n_1,j}. \end{aligned}$$

Thus

$$n_2\cdot l_{n_1,n_2} = \frac{1}{2}\sum_{j=0}^{n_2-2}(-1)^{n_2-j}\, l_{n_1,j} - \frac{1}{2}\, l_{n_1,n_2-1} = -(n_2-1)l_{n_1,n_2-1} - \frac{1}{2}\, l_{n_1,n_2-1}.$$

Note that $l_{0,n_2} = l_{n_2,0}$, we have

$$\begin{aligned} l_{n_1,n_2} &= \frac{-\frac{1}{2}-n_2+1}{n_2} l_{n_1,n_2-1} = \cdots = \binom{-\frac{1}{2}}{n_2} l_{n_1,0} \\ &= \binom{-\frac{1}{2}}{n_2} l_{0,n_1} = \binom{-\frac{1}{2}}{n_2}\binom{-\frac{1}{2}}{n_1} l_{0,0}. \end{aligned}$$

Therefore (i) is valid and (ii) follows form (i). □

5.3 Limits for the hyperbolic metric near corners

When the order $\alpha < 1$, the analogue limit (5.1.2) exists but cannot be described only in terms of the curvature of $\lambda(z)$. Only in few special cases l in (5.1.2) can be given explicitly. In this section we list several results for the hyperbolic metric on $\mathbb{D}^*$ and $\mathbb{C}\backslash\{0,1\}$ with a corner at the origin of order $\alpha \in (0,1)$. The following theorem is due to Kraus, Roth and Sugawa [21]. They did not give the explicit formula of (5.3.7), but it is easy to deduce it from (4.1.2).

Theorem H. *Let $0 < \alpha < 1$. Then for the hyperbolic density $\lambda(z)$ given in* (4.1.2), *we have*

$$\lim_{z\to 0}|z|^{\alpha}\lambda(z) = \frac{\delta}{1-\delta^2}(1-\alpha) \tag{5.3.7}$$

where δ is as in (4.1.5), *a, b and c are as in* (4.1.4).

The following result is evident by Theorem Theorem A.

Theorem 5.3.1. *If $\lambda_\alpha(z)|dz|$ is the hyperbolic metric on $\mathbb{D}^*$ with the order $\alpha \in (0,\,1)$ at the origin, then the limit l defined in* (5.1.2) *satisfies $l = 1-\alpha$.*

For the hyperbolic metric $\lambda_{\alpha,\beta,\gamma}(z)|dz|$ on $\mathbb{C}\backslash\{0,1\}$ when the order $0<\alpha<1$, the following result corresponding to Theorem 5.1.1 holds.

Theorem 5.3.2 ([34]). *For $m,\, n \geq 0$, $0 < \alpha < 1$ and $\lambda(z)$ as in (4.1.1), the limit*

$$l_{m,\,n} := \frac{1}{m!n!} \lim_{z\to 0} |z|^{\alpha} \bar{z}^{m} z^{n} \bar{\partial}^{m} \partial^{n} \lambda(z)$$

exists, and $l_{m,\,n}$ satisfies the following

(i) $l_{m,\,n} = \binom{-\frac{\alpha}{2}}{n}\binom{-\frac{\alpha}{2}}{m} l$,

(ii) $l_{m,\,n} = l_{n,\,m}$,

where

$$l := l_{0,\,0} = \lim_{z\to 0} |z|^{\alpha}\lambda(z) = \frac{\delta}{1-\delta^2}(1-\alpha). \tag{5.3.8}$$

Its proof is similar to Theorem 5.2.1. We end this section with a specific version of Theorem 5.2.1.

Theorem 5.3.3. *For $\lambda(z) := \lambda_{\alpha,\,\beta,\,\gamma}(z)$ as in (4.1.1) with order $\alpha \in (0,\,1)$, let $u(z) := \log\lambda(z)$. Then for $m,\, n \geq 1$,*

(i) $\lim_{z\to 0} z^n \partial^n u(z) = \frac{\alpha}{2}(-1)^n(n-1)! = \lim_{z\to 0} \bar{z}^n \bar{\partial}^n u(z)$,

(ii) $\lim_{z\to 0} \bar{z}^m z^n |z|^{2\alpha-2} \bar{\partial}^m \partial^n u(z) = \frac{(-1)^{n+m}(c)_{n-1}(c)_{m-1}K_2^2}{K_1K_2-1}\left(H(a,b,c)\right)^2$ *where $H(a,b,c)$ is defined by (4.2.7).*

Remark 5.3.4. *The limit in (i) of Theorem 5.3.3 is contained in (i) of Theorem 5.2.1, and was the limit (ii) of Theorem 5.3.3 is stronger than (ii) of Theorem 5.2.1. In what follows we prove it in a different way to show the particularity of $\lambda_{\alpha,\beta,\gamma}(z)|dz|$. The proof of Theorem 5.3.3 is an application of Lemma 4.2.1.*

Proof of Theorem 5.3.3. We note that

$$u(z) = -\alpha\log|z| - \beta\log|1-z| + \log K_3 - \log M(z)$$

with $M(z)$ as in (4.2.6). At first we consider $\partial^n \log M(z)$. From (4.2.10) we know that $M(0) > 0$ for any $a,\ b,\ c$ given by (4.1.4). From Lemma 4.2.1 we have

$$\lim_{z\to 0} z^k \partial^k M(z) = 0$$

for all $k \geq 1$ and $0 < \alpha < 1$. Since $\partial^n \log M(z)$ is a linear combination of products of $\frac{\partial^k M}{M}$ with $k \leq n$, when $n \geq 1$,

$$\lim_{z\to 0} z^n \partial^n \log M(z) = 0.$$

Note that

$$\partial^n \log|z| = \frac{(-1)^{n-1}(n-1)!}{2z^n},$$

combining with (4.2.18) results in the first equality in (i).

For the second equality in (ii), note that $u(z)$ is real-valued, that is

$$\lim_{z\to 0} \bar{z}^n \bar{\partial}^n u(z) = \lim_{z\to 0} \overline{z^n \partial^n u(z)} = \frac{\alpha}{2}(-1)^n(n-1)!.$$

Therefore (i) is valid.

Now we discuss the term $\bar{\partial}^m \partial^n \log M(z)$ to complete the proof. Since $\bar{\partial}^m \partial^n \log M(z)$ is a linear combination of products of $\bar{\partial}^t \partial^k M/M$ with $0 \leq t \leq m$, $0 \leq k \leq n$, Lemma 4.2.1 implies that

$$\lim_{z \to 0} \bar{z}^m z^n |z|^{2\alpha-2} \prod_{j=2}^{N} \frac{\bar{\partial}^{t_j} \partial^{k_j} M(z)}{M(z)} = 0,$$

where $2 \leq N \leq m+n$, $1 \leq t_j \leq m$ and $1 \leq k_j \leq n$ for every index j, $2 \leq j \leq N$. Thus

$$\begin{aligned} & \lim_{z \to 0} \bar{z}^m z^n |z|^{2\alpha-2} \bar{\partial}^m \partial^n \log M(z) \\ = \; & \lim_{z \to 0} \bar{z}^m z^n |z|^{2\alpha-2} \frac{\bar{\partial}^m \partial^n M(z)}{M(z)} \\ = \; & \frac{(-1)^{n+m} (c)_{n-1} (c)_{m-1} K_2^2}{K_1 K_2 - 1} (H(a,b,c))^2. \end{aligned}$$

Note that $\bar{\partial}^m \partial^n \log |1-z| = 0$, $\bar{\partial}^m \partial^n \log |z| = 0$, therefore (ii) holds. □

Acknowledgement

Foremost, I would like to express my sincere gratitude to my supervisor Professor Toshiyuki Sugawa for the continuous support of my doctoral study and research. I am grateful for his patience, motivation, enthusiasm, and immense knowledge. His warm help made me pull through the difficult time after 2011 Tohoku Earthquake. I could not have imagined a better supervisor and mentor than him.

My sincere thanks also go to Doctor Limei Wang for helping me deal with all kinds of procedures when I was new in Sendai, and Doctor Shixin Xu for showing me practical skills in using LaTeX and Mathematica.

I am highly indebted to Doctor Ikkei Hotta, Doctor Hiroshi Yamamoto, Rintaro Ohno, Hirokazu Shimauchi, Naoki Hiranuma, Yan Jiang, Lijie Sun and Ming Li, for their kind help in Japanese lessons, weekly seminars and long-distance discussions, and for all the fun we have had in the last three years. I really enjoyed the movie sessions, the game afternoons, and the snacks brought from China.

I appreciate Ocean University of China and China Scholarship Council, for financial support.

I thank my friends in Japan: Shijia Yao, Zhaowei Ma, Fujie Huang, Doctor Tao Li and Doctor Yanyan Wang. Without their aids, I could have not accustomed myself to the new surroundings in a short time. In particular, I am grateful to Shijia Yao and Fujie Huang for offering me places to stay when I was back to Sendai after the earthquake.

Last but not the least, I would like to thank Bufang and Busan for their listen and company, at any time and place.

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